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SHOCK HYDRODYNAMICS

A SERIES OF LECTURES BY:

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I. RIEMANN METHOD

Lectures by J. von Neumann

In this treatment we shall consider the hydrodynamics of compressible fluids. In the applications in which we are most interested, the motions are so rapid that there is not sufficient time to transfer an appreciable amount of momentum or energy across streamlines. Therefore we are justified in neglecting viscosity and heat conductivity.

Neglecting heat conductivity assumes that $\kappa J\, d^2T/dx^2$ is small compared to $(\rho u^2/g)\, du/dx$. Here κ is the coefficient of heat conductivity, J is the mechanical equivalent of heat, T is the temperature, g is the gravitational constant, and u is the velocity of the fluid. In all of the applications that we are interested in, this approximation is really satisfactory. For example, if iron were accelerated from rest to 10^5 cm/sec velocity in 10 cms, and if the gradient of the temperature gradient were $1000^\circ C/cm^2$, then the heat conduction term would only be 1/500th the value of the kinetic term.

However, it is somewhat more difficult to justify the neglect of viscosity. This corresponds to neglecting $\mu\, d^2u/dx^2$ in comparison with the pressure gradient. Here μ is the usual coefficient of viscosity. If the pressure gradient is 10^5 bars per cm and the gradient of the velocity gradient is 10^4 per sec per cm, then for water neglect of viscosity corresponds to neglecting a term of the order of 200 in comparison to a term of the order of 10^{11} 1).

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1) See Durand, Aerodynamics, vol. III, p. 219, for a discussion by G.I. Taylor of the effect of viscosity and heat conduction on the sharpness of a shock wave.

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For solids instead of viscosity the resistance to plastic flow appears and is sometimes important. However this will not be considered here. (See W. G. Penney, LA-155.) Radiation is neglected because it would introduce a number of additional complications.

1. EULERIAN AND LAGRANGIAN FORM OF THE EQUATIONS

There are two ways of describing a hydrodynamical ensemble. In the Eulerian system, we consider the conditions of pressure, p; density, ρ; temperature, T; etc. of the fluid passing a fixed point in space. In the Lagrangian system, we see how these conditions of the fluid change with time when we follow the motion of the individual particles. Let us derive the one-dimensional equations of motion for the two systems.

(a). Lagrangian Form of Equation of Motion

Each particle is designated by a value of the symbol ζ. Here ζ can correspond with the position of the particle at the time zero, or with any other arbitrary convention. At any time, t, the position of this particle is designated by $x(\zeta, t)$. The motion of a particle must satisfy Newton's equation:

$$M\, d^2x/dt^2 = F \qquad (1)$$

Here, our particle consists of the fluid elements lying between ζ and $\zeta + d\zeta$. The mass of this particle is $M = \rho_0 d\zeta$. The force acting on it in the x direction is the pressure at ζ minus the pressure at $\zeta + d\zeta$, or

$$F = -\,(\partial p/\partial \zeta)\, d\zeta \qquad (2)$$

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And Equation (1) becomes:

$$d^2x/dt^2 = - (1/\rho_o)\ (\partial p/\partial \xi) \qquad \text{(Lagrange)} \qquad (3)$$

Here we are using the usual hydrodynamical convention of letting the total derivative with respect to time mean that we are following the motion of the individual particles.

(b). Eulerian Form of Hydrodynamical Equations

The Eulerian form of the hydrodynamical equations is convenient when we are concerned with the properties of a fluid passing a fixed point. Let $u(x,t)$ be the velocity of the fluid relative to the fixed point. Then $u = dx/dt$ and

$$d^2x/dt^2 = du/dt = \partial u/\partial t + (dx/dt)(\partial u/\partial x) = \partial u/\partial t + u(\partial u/\partial x) \qquad (4)$$

In order to derive the equation of motion, we consider as our particle those fluid elements which lie between x and $x + dx$ at the time t. This particle has the mass, $M = \rho\, dx$. The force acting on this particle in the x direction is the pressure at x minus the pressure at $x + dx$ or $F = -(\partial p/\partial x)\, dx$. Substituting these relations into Newton's equation:

$$\partial u/\partial t + u\ (\partial u/\partial x) = - (1/\rho)\ (\partial p/\partial x) \qquad \text{(Euler)} \qquad (5)$$

The equation of continuity can be derived in the following manner. Consider the fluid entering and leaving a little element of volume lying between x and $x + dx$. In a length of time, dt, the mass of material entering from the left is $\rho u\, dt$ and the material leaving from the right is $\left\{ \rho u + \left[\partial(\rho u)/\partial x\right] dx \right\} dt$. Thus the net increase of material in this element, $(\partial\rho/\partial t)\, dx\, dt$, is $- \left[\partial(\rho u)/\partial x\right] dx\, dt$ and the equation of continuity is

$$\partial\rho/\partial t = - \partial(\rho u)/\partial x \qquad (6)$$

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A third equation that we require is the conservation of energy. The work which is done in unit time on the fluid flowing between x and $x + dx$ is equal to the pressure times the velocity at x minus the pressure times the velocity at $x + dx$ or $-(\partial/\partial x)(up)\,dx$. Therefore the work done on a unit volume of gas in unit time is $-(\partial/\partial x)(up)$. However a unit mass of material occupies a specific volume, $V = 1/\rho$. Thus the work done on a unit mass of material in unit time is $-V\,(\partial/\partial x)(up)$. By the conservation of energy, this work must be equal to the rate of change of kinetic plus internal energy of a unit mass of material. The energy of the system, E (per unit mass) is given by the relation:

$$E = (1/2)u^2 + E_{int}$$

Thus the equation of conservation of energy is expressed by

$$(d/dt)\left((1/2)u^2 + E_{int}\right) = \left(\partial/\partial t + u(\partial/\partial x)\right)\left((1/2)u^2 + E_{int}\right) = -V\,(\partial/\partial x)\,(up) \qquad (7)$$

2. BEHAVIOUR OF ENTROPY, RELATION WITH MECHANICS, THERMODYNAMICS, AND IRREVERSIBILITY

In addition to the equation of motion and equation of continuity, we have the equation of state and the equation of conservation of energy. These four equations should be sufficient to determine the four variables p, ρ, T, and u as functions of position and time. However, there was considerable confusion up to a very few years ago as to whether the fourth equation should be the conservation of energy or the constancy of entropy. As we shall show, as long as the fluid motion involves no abrupt changes in

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pressure or velocity, the conservation of energy implies the constancy of entropy and vice versa. But whenever an abrupt change, or shock wave, occurs the conservation of energy leads to a definite change in the entropy.

(a). Constancy of Entropy with Time for a Fluid Element (Assuming No Shock Waves).

In fluids whose elements remain in thermodynamical equilibrium during the motion, changes in state are reversible and the entropy of a fluid element will remain constant with time. The fluid elements do not remain in equilibrium during the motion in which the fluid passes through a shock or when the motion is too rapid to maintain either chemical equilibrium or to maintain equilibrium in the rotational and vibrational degrees of freedom of the molecules (see Lewis and Von Elbe, Combustion, Flames, and Explosions of Gases, Cambridge, 1938). The fact that entropy is conserved whenever the changes in state are reversible may be verified in the following manner. The internal energy can be expressed in terms of the specific volume and the entropy, S (per unit mass). Thus the equation of conservation of energy (7) becomes:

$$\left((\partial/\partial t) + u(\partial/\partial x)\right)\left(\tfrac{1}{2}u^2 + E_{int}\right) = -V(\partial/\partial x)(up) \qquad (8)$$

and, carrying out the indicated operations:

$$u\left[(\partial u/\partial t) + u(\partial u/\partial x)\right] + (\partial E_{int}/\partial V)_S \left[(\partial V/\partial t) + u(\partial V/\partial x)\right]$$

$$+ (\partial E_{int}/\partial S)_V \left[(\partial S/\partial t + u(\partial S/\partial x)\right] = -Vu(\partial p/\partial x) \qquad (9)$$

$$- Vp(\partial u/\partial x)$$

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But from the equation of motion (5)

$$u\left[(\partial u/\partial t) + u(\partial u/\partial x)\right] = - V u\ (\partial p/\partial x) \tag{10}$$

And from the equation of continuity (6)

$$\begin{aligned} \partial V/\partial t + u(\partial V/\partial x) &= - 1/\rho^2 \left[\partial \rho/\partial t + u(\partial \rho/\partial x)\right] \\ &= 1/\rho\ (\partial u/\partial x) = V\ (\partial u/\partial x) \end{aligned} \tag{11}$$

Then remembering that for a reversible change the internal energy is the usual energy, E, of thermodynamic systems (Thermodynamics, Lewis and Randall, McGraw-Hill, 1923, see page 164), it must have the properties:

$$\left(\frac{\partial E_{int}}{\partial V}\right)_S = - p \tag{12}$$

and

$$\left(\frac{\partial E_{int}}{\partial S}\right)_V = T \tag{13}$$

Thus Eq. (9) becomes:

$$- Vu \frac{\partial p}{\partial x} - pV \frac{\partial u}{\partial x} + T\left[\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x}\right] = - Vu \frac{\partial p}{\partial x} - Vp \frac{\partial u}{\partial x} \tag{14}$$

or

$$T\left[\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x}\right] = T \frac{dS}{dt} = 0 \tag{15}$$

From Equation (15) it is apparent that the entropy of each particle does not change with time. If the entropy had the same value throughout the whole fluid at any time, it must maintain this value for all subsequent times.

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3. RIEMANN'S METHOD OF INTEGRATION (ONE DIMENSIONAL AND ISENTROPIC).

Riemann developed a very useful method of integrating the equations of motion for one-dimensional isentropic flow problems (see Riemann's Collected Papers, Durand's Aerodynamics, Vol. III., or the first edition of Riemann-Weber).

Let us suppose that initially, the specific entropy throughout the fluid is a constant, S_o. Then, from the theorem of the last section, we know that for all subsequent time the specific entropy of the system remains S_o. (This is not true after a shock wave has developed, but we shall consider such cases later.) Since the entropy is constant, we can write the equation of state in the form of the adiabat:

$$p = P(V, S_o) \tag{16}$$

The equation of motion (5) becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -V\frac{\partial p}{\partial x} = -V\left(\frac{\partial P}{\partial V}\right)_{S_o}\frac{\partial V}{\partial x} \tag{17}$$

And the equation of continuity (6) is:

$$\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} = V\frac{\partial u}{\partial x} \tag{18}$$

If we consider any function $\sigma\,(V, S_o)$, then by virtue of Eq. (18) and the constancy of entropy throughout the system,

$$\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right]\sigma = \left(\frac{\partial \sigma}{\partial V}\right)_{S_o}\left[\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x}\right] = V\left(\frac{\partial \sigma}{\partial V}\right)_{S_o}\frac{\partial u}{\partial x} \tag{19}$$

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Adding together Eqs. (17) and (19):

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)(u + \sigma) = -V\left(\frac{\partial P}{\partial V}\right)_{S_0}\frac{\partial V}{\partial x} + V\left(\frac{\partial \sigma}{\partial V}\right)_{S_0}\frac{\partial u}{\partial x} \tag{20}$$

Riemann's trick[2] was to choose σ so that

$$-\left(\frac{\partial P}{\partial V}\right)_{S_0}\frac{\partial V}{\partial x} = \left(\frac{\partial \sigma}{\partial V}\right)_{S_0}\frac{\partial \sigma}{\partial x} \tag{21}$$

For in this case the right-hand side of the equation becomes simply

$$V\left(\frac{\partial \sigma}{\partial V}\right)_{S_0}\frac{\partial}{\partial x}(u + \sigma) \tag{22}$$

This is accomplished by letting

$$\left(\frac{\partial \sigma}{\partial V}\right)_{S_0} = -\sqrt{-\left(\frac{\partial P}{\partial V}\right)_{S_0}} = -\sqrt{\frac{1}{V^2}\left(\frac{\partial P}{\partial \rho}\right)_{S_0}} = -\frac{c}{V} \tag{23}$$

Here $c(V, S_0)$ is the local velocity of sound

$$c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{S_0}} \tag{24}$$

The value of σ itself is obtained by integrating Eq. (23)

$$\sigma = \int_V^{V_0}\frac{c}{V}\,dV = \int_{\rho_0}^{\rho}\sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S}\,\frac{d\rho}{\rho} \tag{25}$$

From the equation of state of the adiabat, both c and σ may be determined as functions of V.

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2) See G. I. Taylor's article in Durand's Aerodynamics, Volume III, p. 215.

[redacted]

Restricting ourselves to this definition of σ, Eq. (20) becomes:

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right)(u + \sigma) = + V \left(\frac{\partial \sigma}{\partial V}\right)_{S_0} \frac{\partial}{\partial x} (u + \sigma) = -c \frac{\partial}{\partial x} (u + \sigma) \tag{26}$$

Or transposing

$$\left[\frac{\partial}{\partial t} + (u + c) \frac{\partial}{\partial x}\right] (u + \sigma) = 0 \tag{27}$$

Similarly if we had subtracted Eq. (19) from (17) we would have obtained the relation:

$$\left[\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right] (u - \sigma) = 0 \tag{28}$$

These equations have a simple interpretation provided that we change our frame of reference. Instead of observing the conditions of the fluid at a fixed point as in the Eulerian system, or following the motion of the individual particles as in the Lagrangian system, we now observe the changes which take place in the fluid when our frame of reference moves with the local velocity of sound with respect to the moving fluid. In order to move with the velocity of sound in the minus x direction, our frame of reference must have the velocity $u - c$. In order to move with the velocity of sound in the x direction, our frame of reference must have the velocity $u + c$.

Equation (27) states that if we start at any point in the fluid and move with a velocity $u + c$, we will find that the quantity $u + \sigma$ remains constant.

Equation (28) states that if we start at any point in the fluid and move with the velocity $u - c$, we will find that the quantity $u - \sigma$ remains

[redacted]

constant.

Thus if we know the velocity and density at all points in the fluid at some time, these two equations serve to define the values of $u - \sigma$ and $u + \sigma$ at any subsequent time. And knowing the values of $u - \sigma$ and $u + \sigma$, we also know the values of u and σ separately. From σ and the adiabatic equation of state, we can determine the density and the pressure. This in principle forms a complete solution to the problems of one-dimensional isentropic flow.

It should be emphasized that the Riemann method is only applicable to one-dimensional problems and cannot be generalized to two or three dimensions.

4. DISCUSSION OF SOME EXAMPLES WITH RIEMANN'S METHOD

(a). Method of Numerical Integration in General Case

The Riemann method can be used in the following manner to integrate numerically the equations of motion. Suppose that at the time $t = 0$ we are given the velocity, u, and the specific volume, V, at a set of points x_1, x_2, ... x_n. We are also given the equation of the adiabat. We proceed as follows:

First, we use the equation for the adiabat to calculate

$$c(V,S_o) = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{S_o}}$$

and then calculate

$$\sigma(V,S_o) = \int_V^{V_o} \frac{c(V,S_o)}{V}\, dV$$

From $V(x_i,\ t = 0)$ and the above relationships, we determine

$$\left.\begin{array}{l} c(x_i,\ t = 0) \\ \sigma(x_i,\ t = 0) \end{array}\right\} \quad i = 1,\ 2,\ \ldots,n$$

[redacted]

From $u(x_i, t = 0)$ and the above we determine the two sets of numbers

$$f_i = u(x_i, t = 0) + \sigma(x_i, t = 0) \qquad i = 1, 2, \ldots, n$$
$$g_i = u(x_i, t = 0) - \sigma(x_i, t = 0)$$

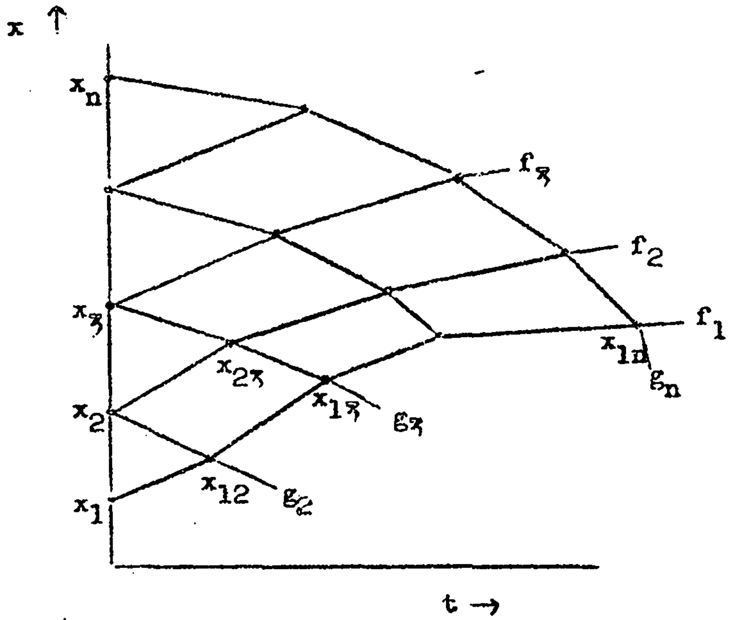

Fig. 1

In the Riemann method we try to construct the lines along which $n + \sigma = f_i$ and $n - \sigma = g_i$. Figure 1 shows what such a mesh might look like when we integrate numerically. It is easy to find the points of intersection graphically. From each point, x_i, we draw two lines one with the slope $u(x_i, t = 0) + \sigma(x_i, t = 0)$ and the other line with the slope $u(x_i, t = 0) - \sigma(x_i, t = 0)$. Along the first line f_i remains constant, along the second g_i remains constant. The places where these lines intersect forms the points x_{12}, x_{23}, ... At these intersections, we know the value of f and g.

[redacted]

At x_{12}: $$u(x_{12}, t_{12}) = \frac{1}{2}(g_2 + f_1)$$

$$\sigma(x_{12}, t_{12}) = \frac{1}{2}(-g_2 + f_1)$$

At x_{23}: $$u(x_{23}, t_{23}) = \frac{1}{2}(g_3 + f_2)$$

$$\sigma(x_{23}, t_{23}) = \frac{1}{2}(-g_3 + f_2)$$

Knowing the values of σ at the intersection points, we can determine the corresponding values of V and of c. Then using these new values of c and u we can again draw two lines through each intersection point and determine a new set of intersection points, etc. And in this way we can carry out the whole integration.

It is interesting to notice that by this method we cannot obtain any information about the fluid motion outside of the rough triangle bounded by the line $f = f_1$ and by the line $g = g_n$. The conditions of the fluid motion within this triangle are completely unaffected by the conditions of the fluid outside of the triangle.

(b). Application of Riemann's Method to Ideal One-dimensional Gas.

The Riemann Method is particularly useful when the fluid satisfies the ideal gas form of adiabat[3]:

$$p = k(S_o)\rho^\gamma \tag{29}$$

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3) For many applications it is useful to take the adiabat in the form:

$$p = k'(S_o)\rho^\gamma + p_o(S_o)$$

This does not change the resultant fluid motions since the hydrodynamical equations only involve pressure differences.

For other purposes,

$$p(V - b)^\gamma = k''(S_o)$$

is a useful form.

[redacted]

Then, since

$$\left(\frac{\partial p}{\partial \rho}\right)_{S_0} = \gamma\, k(S_0)\rho^{\gamma-1} \tag{30}$$

$$c = \sqrt{\gamma k}\;\rho^{(\gamma-1)/2} = \frac{\sqrt{\gamma k}}{V^{(\gamma-1)/2}} \tag{31}$$

And

$$\sigma = \int_V^{V_0} \frac{c}{V}\,dV = \sqrt{\gamma k}\int_V^{V_0} \frac{dV}{V^{(\gamma+1)/2}} = \sqrt{\gamma k}\left(\frac{2}{\gamma-1}\right)\left[\frac{+1}{V^{(\gamma-1)/2}} - \frac{1}{V_0^{(\gamma-1)/2}}\right] \tag{32}$$

Since V_0 is arbitrary, it is convenient to set $V_0 = \infty$. The value chosen for V_0 cannot affect any physical properties of the fluid. Then

$$\sigma = +\left(\frac{2}{\gamma-1}\right)c \tag{33}$$

If γ should equal 3, there are many simplifications which appear. In this case, the velocity of sound is proportional to the density, $\frac{2}{\gamma-1} = 1$ and $\sigma = +c$. Most substances under very high pressure approximately follow the ideal-gas adiabat[4] with $\gamma = 3$. Under these conditions $f = u + c$ is constant along the curve whose slope is $u + c$, and $g = u - c$ is constant along the curve whose slope is $u - c$. Thus f is constant along the line:

$$x = a(f) + ft \tag{34}$$

4) This value of γ is not to be confused with the true ratio of specific heats, γ', which varies between 5/3 and 1. For example any substance satisfying the equation of state $pV^{3/\gamma'} = aT^{3(\gamma'-1)/2\gamma'}$ has the adiabat, $p = (\text{const})\,\rho^3$.

[redacted]

And g is constant along the line

$$x = b(g) + gt \tag{35}$$

Solving these equations simultaneously:

$$t = \frac{b(g) - a(f)}{-g + f} \tag{36}$$

$$x = \frac{f\,b(g) - g\,a(f)}{-g + f} \tag{37}$$

$$u = \frac{1}{2}(g + f) \tag{38}$$

$$c = \frac{1}{2}(g - f) \tag{39}$$

These four equations form a complete parametric solution to the equations of motion. If we know the velocities and densities at the time $t = 0$, we can determine $a(g)$ and $b(f)$. Knowing $a(g)$ and $b(f)$, we can solve (36) and (37) simultaneously to determine the value of g and f for any desired value of x and t. Knowing g and f we can use Eqs. (38) and (39) to determine u and c. Then substituting c into Eq. (31) we get V and hence all of the properties of the fluid motion.

Darboux obtained analytical solutions to the equations of motion for all values of γ such that $\gamma = \frac{2m+1}{2m-1}$ where m is an integer. These values lie in the useful range of γ. The second and third are of practical interest representing very accurately an ideal monotomic gas and air, respectively.

m	γ
1	3
2	1.667
3	1.400
4	1.286
5	1.222

[redacted]

 [redacted]

In his solutions for x and t, Darboux obtained expressions involving derivatives of a and b up to the order $m - 1$.

(c) Riemann Method Applied to Disturbance Coming From One Direction

The Riemann method is particularly easy when a disturbance comes from one direction. Let us suppose that at time, $t = 0$, the fluid at points such that x is positive is at rest and the fluid corresponding to negative values of x is disturbed. In this case we shall show that the lines of constant $u + \sigma$ are straight lines and all along these lines the values of u and σ separately remain constant. We suppose that the fluid satisfies the ideal gas adiabat so that

$$\sigma = \frac{2}{\gamma - 1} c \tag{40}$$

Figure 2 illustrates the problem.

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5) These equations of Darboux are given in Hadamard, Lecons sur la Propagation des Ondes (Paris, 1902). They amount to the simultaneous solutions of the following parametric equations:

$$Z = \left(\frac{1}{\sigma} \frac{\partial}{\partial \sigma}\right)_u^{m-1} \left[\frac{b(u - \sigma) + a(u + \sigma)}{\sigma}\right]$$

$$t = \left(\frac{\partial Z}{\partial u}\right)_\sigma$$

$$x = -\sigma\left(\frac{\partial Z}{\partial \sigma}\right)_u$$

Here as before a is an arbitrary function of $u + \sigma$ and b is an arbitrary function of $u - \sigma$.

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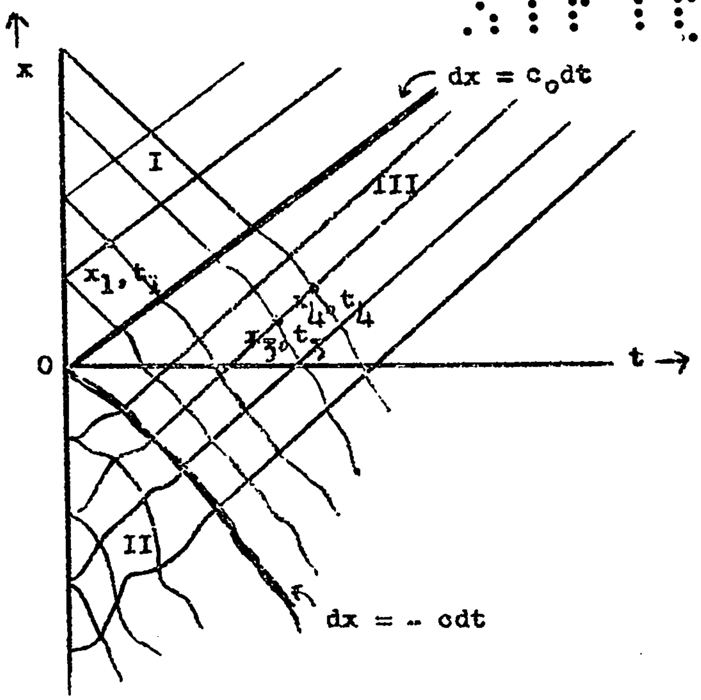

Figure 2

DISTURBANCE EMANATING FROM LEFT

We must distinguish three regions. In region I, the fluid is undisturbed. In region II, the disturbance is coming from both directions. In region III, the disturbance is only coming from the negative x direction.

Region I is bounded by the line $dx = c_0 dt$. In region I, $u = 0$, $c = c_0$, and $\sigma = \frac{2}{\gamma - 1} c_0$.

At any intersection of points in region I such as (x_1, t_1):

$$u - \sigma = - \frac{2}{\gamma - 1} c_0 \tag{41}$$

$$u + \sigma = + \frac{2}{\gamma - 1} c_0 \tag{42}$$

So that u remains zero and σ remains $\frac{2}{\gamma - 1} c_0$. The lines of constant $u - \sigma$ and constant $u + \sigma$ in this region, as constructed in the Riemann method, are therefore straight lines.

[redacted]

In region II, the disturbance may be quite general.

In region III, the lines of constant $u + \sigma$ must be straight lines since all of the lines of constant $u - \sigma$ which they intersect have the same characteristic value, $u - \sigma = - \frac{2\,c_o}{\gamma_o - 1}$. Thus at any point (x_3, t_3) along the line of slope $u + c$ characterized by $u + \sigma = f_i$ we have:

$$u + \sigma = f_i \tag{43}$$

$$u - \sigma = - \frac{2\,c_o}{\gamma - 1} \tag{44}$$

$$u = \frac{1}{2}\left(f_i - \frac{2\,c_o}{\gamma - 1}\right) \tag{45}$$

$$\sigma = \frac{1}{2}\left(f_i + \frac{2\,c_o}{\gamma - 1}\right) = \frac{2\,c}{\gamma - 1} \tag{46}$$

The line for which $u + \sigma = f_i$, therefore has the slope:

$$\begin{aligned} u + c &= \frac{1}{2}\left(f_i - \frac{2\,c_o}{\gamma - 1}\right) + \frac{\gamma - 1}{4}\left(f_i + \frac{2\,c_o}{\gamma - 1}\right) \\ &= \frac{\gamma + 1}{4} f_i + \frac{\gamma - 3}{2(\gamma - 1)}\, c_o \end{aligned} \tag{47}$$

Since this slope does not change throughout region III, it follows that the lines of constant $u + \sigma$ are straight lines. It is important to note that the lines for different values of $u + \sigma = f_i$ have different slopes. If two of these lines approach or intersect each other, the pressure gradients become large and then infinite and a shock wave occurs. The Riemann method is no longer applicable when these lines intersect.

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In region III, the lines characteristic of $u - c$ have the slope

$$
\begin{aligned}
u - c &= \frac{1}{2}\left(f_i - \frac{2\,c_0}{\gamma - 1}\right) - \frac{\gamma - 1}{4}\left(f_i + \frac{2\,c_0}{\gamma - 1}\right) \\
&= -\frac{\gamma - 3}{4}\,f_i - \frac{\gamma + 1}{2(\gamma - 1)}\,c_0
\end{aligned}
\tag{48}
$$

Since these lines cross lines having different values of f_i, it follows that the lines of characteristic $u - c$ may be curved in passing through region III.

(d) Formation of Shock Waves (Problem of a Moving Wall)

The formation of shock waves occurs quite generally as the result of any disturbance in a gas. If the disturbance is mild, it takes a very long time for the shock wave to develop; if the disturbance is violent, the shock wave forms in a short time. The Riemann method can be used to show how they originate.

Consider a wall or piston which is set into motion at time $t = 0$ and propagates a disturbance in the gas in front of it. The position of the wall at any time is given by the relation,

$$
\begin{aligned}
x_{wall} &= 0 \qquad & t &< 0 \\
&= W(t) \qquad & t &\geq 0
\end{aligned}
$$

For the sake of simplicity, let us restrict the motion of the piston to subsonic velocities. In this case only the lines of slope $u + c$ can come from

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the wall. Figure 3 illustrates the problem. The lines of constant $u + \sigma$ are shown.

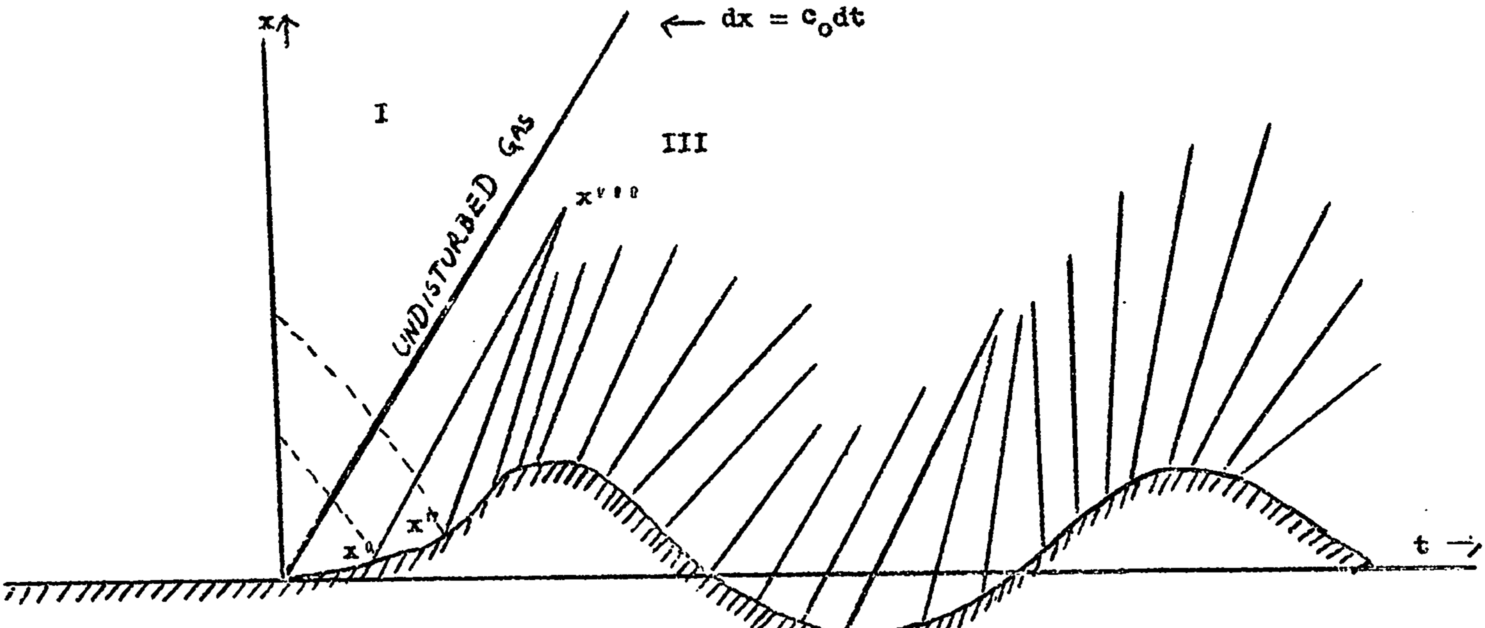

Figure 3

Initially the gas is at rest and has a velocity $u = 0$ as well as a constant velocity of sound, $c = c_0$. We suppose that the gas satisfies the ideal-gas adiabat so that $\sigma = \dfrac{2c}{\gamma - 1}$. As in the previous problem we can construct the line $dx = c_0\, dt$. To the left of this line, the gas remains undisturbed. On the surface of the wall $u = \dfrac{dW}{dt}$. At any point on the surface of the wall, $u - \sigma = -\dfrac{2\, c_0}{\gamma - 1}$, since the lines of constant $u - \sigma$ arise in the undisturbed part of the fluid. Thus on the surface of the wall:

$$\sigma = \frac{2\, c_0}{\gamma - 1} + \frac{dW}{dt} \tag{49}$$

[redacted]

$$c = c_o + \frac{\gamma - 1}{2} \frac{dW}{dt} \qquad (50)$$

$$u + c = c_o + \frac{\gamma + 1}{2} \frac{dW}{dt} \qquad (51)$$

Therefore, whenever $\frac{dW}{dt}$ is positive, $c > c_o$ and the density of the gas in the neighborhood of the wall is increased, if $\frac{dW}{dt}$ is negative, the gas in the neighborhood of the wall is attenuated.

The fluid lying to the right of the line $dx = c_o\, dt$ and above the wall corresponds to the fluid in region III of the last problem. We therefore know that the lines of constant $u + c$ emanating from the wall are straight lines. A line starting from a point, (x', t'), where $\frac{dW}{dt}$ is positive and small will have a slope, $u + c$, less than the slope of a line starting from a point, (x'', t''), where $\frac{dW}{dt}$ is larger. These two lines must therefore meet at some point (x''', t'''). Since these two lines have different velocities of sound, they also have different densities. So as they approach each other a progressively sharper density gradiant develops. This gives rise to a shock wave and under such conditions the Riemann method is no longer valid. From the above, it is clear that the less the piston or wall is accelerated, the smaller will be the difference in slopes of the lines of constant $u + c$ and the longer time it will take for the lines to come together to form large density gradients and shock waves.

From Figure 3 and Eq. (51) it is clear that:

1. Whenever the piston is accelerating the lines of constant u tend to come together to form shock waves.

2. Whenever the piston is decelerating the lines of constant u

[redacted]

tend to go apart and not form shock waves.

These are special cases of the more general theorem that shock waves tend to be formed (always will be, if given sufficient time) when a gas is compressed but not when it is being rarefied. In order to get shock waves, it is not necessary for there to be a discontinuity of the motion of the wall.

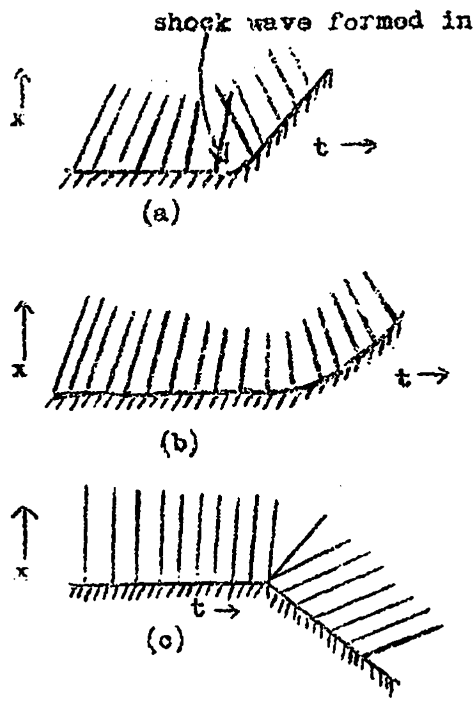

Figure 4

Consider the following examples illustrated in Figure 4.

(a) Push piston into gas suddenly. Get shock wave immediately at wall.

(b) Push piston into gas gradually. Get shock wave later.

(c) Withdraw piston suddenly from gas. No shock wave is formed. Pressure and density gradients get less steep as you go into gas.

[redacted]

II. SHOCK WAVES AND DISCONTINUITIES (One-Dimensional)

Lecture by J. von Neumann

(6). HUGONIOT'S EQUATIONS FOR SHOCKS.

There are two different types of discontinuities in a fluid. In the first kind, there is no flow across the boundary and there is no pressure difference on the two sides of the boundary. In this case, the boundary is just a streamline separating two phases of fluid which may be made up of different chemical substances or the same substance but having different temperatures and densities on the two sides of the boundary, etc. However, the kind of discontinuity in which we are most interested involves the flow of material across a boundary in which a sharp change in pressure, density, and velocity take place. These are called shock waves or detonations depending on whether the equation of state of the material remains unchanged or whether chemical reactions take place.

Let us postulate the existence of a plane shock wave and examine the conditions of its propagation[6]. First we must define the following quantities:

U = velocity of shock wave

$D_1 + U$ = velocity of matter before passing through shock wave

$D_2 + U$ = velocity of matter after passing through shock wave

ρ_1, ρ_2 = density of fluid before and after passing through shock wave

V_1, V_2 = specific volume of fluid before and after passing through shock wave

P_1, P_2 = pressure before and after passing through shock wave

E_1, E_2 = specific internal energy before and after passing through shock wave.

M = mass of material per unit cross-sectional area flowing through the shock wave in unit time.

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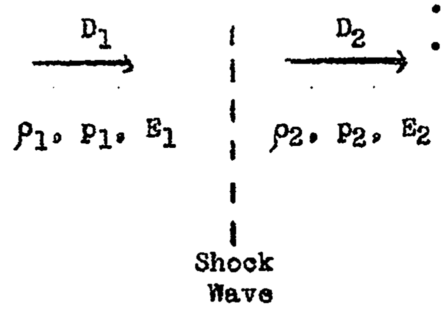
6) G.I. Taylor's article in Durand's <u>Aerodynamics</u> (Springer 1935), Vol. III, page 216 has an excellent discussion of this topic.

[redacted]



In order that this discontinuity can exist we must satisfy the following equations:

(1). conservation of matter

$$D_1\rho_1 = D_2\rho_2 = M \tag{52}$$

(2). conservation of momentum

$$M(D_2 - D_1) = p_1 - p_2 \tag{53}$$

This arises from Newton's equation. Consider the mass of fluid, M, passing through the shock wave per unit area and unit time as forming a particle. Here $P_1 - P_2$ is the force pushing the particle through the shock wave and $M(D_2 - D_1)$ is the rate of change of momentum of the particle.

(3). conservation of energy

$$+ M\,(D_2^2/2 + E_2) - M(D_1^2/2 + E_1) = D_1p_2 - D_2p_2' \tag{54}$$

This equation simply states that the work which is done on the fluid per unit are and time, i. e. $D_1p_1 - D_2p_2$, is equal to the rate of change in its energy. Here $MD^2/2$ is the kinetic energy and ME is the internal energy of the fluid passing through the shock wave. Because of the equation of conservation of momentum, we would get nothing different if we considered the absolute velocity of the fluid rather than its velocity relative to the shock wave.

[redacted]

[redacted]

From the equations for conservation of mass and for momentum, Eqs. (52) and (53) it follows that

$$M = (p_1 - p_2)/(D_2 - D_1) = (p_1 - p_2)/(M/\rho_2 - M/\rho_1) \qquad (55)$$

and

$$M = \pm\sqrt{-(p_1 - p_2)/(1/\rho_1 - 1/\rho_2)} = \pm\sqrt{-(p_1 - p_2)/(V_1 - V_2)}$$

$$= \pm\sqrt{\rho_1\rho_2 \;(p_2 - p_1)/(\rho_2 - \rho_1)} \qquad (56)$$

For weak shocks $M = \rho c$ where $c = \sqrt{(\partial p/\partial \rho)_S}$

From the equations of conservation of matter and energy, Eqs. (52) and (54):

$$D_2^2/2 + E_2 - D_1^2/2 - E_1 = D_1p_1/M - D_2p_2/M = p_1/\rho_1 - p_2/\rho_2 =$$

$$p_1V_1 - p_2V_2 \qquad (57)$$

But by virtue of the equations for conservation of momentum and matter

$$(1/2)(D_2^2 - D_1^2) = (1/2)(D_2 - D_1)(D_2 + D_1) = \left[(p_1 - p_2)/(2M)\right](D_2 + D_1)$$

$$= \left[(p_1 - p_2)/2\right](1/\rho_2 + 1/\rho_1) = (1/2)(p_1 - p_2)(V_2 + V_1) \qquad (58)$$

Thus Eq. (57) becomes

$$(1/2)(p_1 - p_2)(V_2 + V_1) + E_2 - E_1 = p_1V_1 - p_2V_2 \qquad (59)$$

or rearranging

[redacted]

[redacted]

$$E_2 - E_1 = p_1V_1 - p_2V_2 - (1/2)p_1V_1 - (1/2)p_1V_2 + (1/2)p_2V_1 + (1/2)p_2V_2$$

$$= (1/2)p_1V_1 - (1/2)p_2V_2 - (1/2)p_1V_2 + (1/2)p_2V_1 \qquad (60)$$

$$= (1/2)(p_1 + p_2)(V_1 - V_2)$$

And therefore

$$(p_1 + p_2)/2 = (E_2 - E_1)/(V_1 - V_2) \qquad (61)$$

For weak shocks $p = -(\partial E/\partial V)_S$ and the entropy on both sides of the shock wave becomes asymptotically equal.

(7). BEHAVIOR OF ENTROPY. INTERPRETATION. THE RAYLEIGH-TAYLOR THEORY

The characteristics of shock waves can be seen more clearly if we consider the special case of an ideal gas. In this case:[7)]

$$E = pV/(\gamma - 1) \qquad (62)$$

Therefore Eq.(61) becomes:

$$p_1 + p_2 = \left(\frac{2}{\gamma - 1}\right)\frac{p_2V_2 - p_1V_1}{V_1 - V_2} \qquad (63)$$

Or rearranging:

$$\frac{V_1}{V_2} = \frac{(\gamma-1)p_1 + (\gamma+1)p_2}{(\gamma+1)p_1 + (\gamma-1)p_2} \qquad (64)$$

It is convenient to let:

$$p_2/p_1 = \xi \qquad (65)$$

$$\rho_2/\rho_1 = V_1/V_2 = \eta \qquad (66)$$

---

7) For an ideal gas, $pV = RT$, $C_p - C_v = R$, and $C_p/C_v = \gamma$

It follows that: $E = C_vT = \left[C_v/(C_p - C_v)\right] pV = pV/(\gamma - 1)$

[redacted]

Thus Eq. (64) becomes:

$$\boxed{\eta = \left[(\gamma - 1) + (\gamma + 1)\xi\right] / \left[(\gamma + 1) + (\gamma - 1)\xi\right]} \qquad (67)$$

This equation is called the Hugoniot shock adiabatic, although this is obviously a misnomer because entropy is changed in passing through the shock.

If the shock were weak, ξ would be almost unity. It is interesting to expand Eq. (67) for η in powers of $(\xi - 1)$:

$$\eta = 1 + (1/\gamma)(\xi - 1) - (1/2\gamma)(1 - 1/\gamma)(\xi - 1)^2 + (1/4\gamma)(1 - 1/\gamma)^2(\xi - 1)^3 - \ldots\ldots \qquad (68)$$

This series for η agrees through the term in $(\xi - 1)^2$ with the corresponding series expansion which we would get for the compression ratio, $\eta_{\text{no shock}}$, if we allowed the fluid to pass gradually from the region of pressure p_1 to pressure p_2:

$$\eta_{\text{no shock}} = \xi^{1/\gamma} = 1 + (1/\gamma)(\xi - 1) - (1/2\gamma)(1 - 1/\gamma)(\xi - 1)^2 + (1/6\gamma)(1 - 1/\gamma)(2 - 1/\gamma)(\xi - 1)^3 - \ldots\ldots \qquad (69)$$

Since Eq. (69) corresponds to the adiabatic with no change in entropy, it is clear that some entropy change must take place in a shock. The fact that Eqs. (68) and (69) agree so well corresponds to the fact that very little entropy change takes place in a mild shock.

For violent shock waves, where $p_2/p_1 = \xi$ is large, the compression ratio, $\eta = \rho_2/\rho_1$ approaches a constant value:

$$(\eta)_{\text{large } \xi} = (\gamma + 1)/(\gamma - 1) \qquad (70)$$

This limiting value becomes larger as γ becomes smaller. This is seen in the following table:

γ	1	1.2	1.4	1.67	3.0
$(\eta)_{\text{large}\ \xi} = (\gamma + 1)/(\gamma - 1)$	∞	11	6	4	2

The fact that the compression ratio cannot exceed a fixed value in passing through a shock, is quite contrary to the behavior of fluid which pass through the same pressure drop without passing through shocks.

The entropy of an ideal gas can be written in the form [8]

$$S = C_v \log (pV^{\gamma}) + s_o \tag{71}$$

Therefore the change in entropy in passing through the shock wave can be written:

$$\Delta S = S_2 - S_1 = C_v \left[\log(p_2V_2^{\gamma}) - \log (p_1V_1^{\gamma})\right]$$

$$= \gamma C_v \log (\xi^{1/\gamma}\eta^{-1}) \tag{72}$$

The change of entropy is always positive if the fluid flows from a region of low density into a region of higher density, i.e. η is greater than unity. For this case ξ is also greater than unity and the fluid flows from a region of low pressure to a region of greater pressure. For weak shocks, the change in entropy is very small as we can see by expanding Eq. (72) with the help of Eqs. (68) and (69).

$$\Delta S = \gamma C_v \log \left[1 + (1/12\gamma) (1 - 1/\gamma^2) (\xi - 1)^3 + \dots\right]$$

$$= (C_v/12) (1 - 1/\gamma^2) (\xi - 1)^3 + \dots \tag{73}$$

This entropy change is negligible unless $\xi > 2$.

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8) P.S. Epstein, "Thermodynamics" (John Wiley, 1937), p. 63, Eq. 4.19

[redacted]

The problem of shock waves was most confusing to the early workers in this field because of the undertainty as to whether entropy should be conserved in the shock. We now know that it is energy which must be conserved, and we are not greatly concerned over the fact that the entropy changes. However it is always necessary for the fluid to flow through the shock wave in such a direction as to increase entropy. This means that in flowing through a shock wave, MATTER FLOWS FROM A REGION OF LOW DENSITY TO A REGION OF HIGHER DENSITY. NO SHOCK WAVES ARE POSSIBLE WHEN MATTER FLOWS FROM A DENSE TO A LESS DENSE REGION, since this would cause a decrease in entropy. After passing through a shock wave, the fluid becomes hotter than it would if it had arrived at the same pressure without passing through a discontinuity.

Shocks are connected with the nonlinear character of the hydrodynamical equations. In simple physical terms, they may be attributed to the fact that the velocity of sound is not a constant, but increases with the pressure. Suppose we produce a pressure wave moving in the x direction. This situation is shown in Fig. 5. The regions of high pressure at the top of the wave travel with a velocity greater than the velocity in the pressure troughs. The front of the wave gradually gets steeper and the back of the wave gets less steep. After sufficient time, the wave front gets infinitely steep and a true mathematical discontinuity is present.

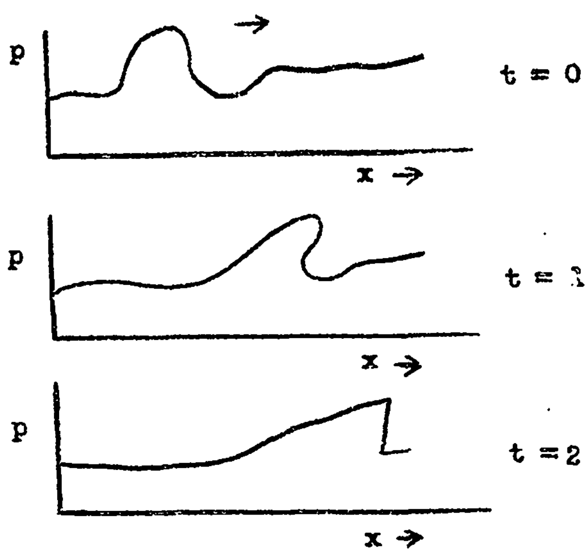

FIGURE 5

[redacted]

Shock waves were discovered in 1860 by Riemann; they were rediscovered in 1890 by Hugoniot and forgotten again. Finally in 1910 Lord Rayleigh and G. I. Taylor started the investigation which has led to our present treatment of the subject. They were very much concerned over the appearance of a mathematical discontinuity in the fluid motion. They made a careful study of the conditions in the fluid in the neighborhood of the shock wave. When they no longer neglected heat conduction and viscosity, they obtained a finite width for the shock wave in which the pressure and density of the fluid changes very rapidly but not discontinuously. A mild shock wave, such as in front of the nose of a bullet, has a width of the order of 10^{-4} cm. For a violent shock, the width is much less.

(8). COLLISIONS BETWEEN GAS MASSES

One of the best examples of one-dimensional shocks is the collision between two gas masses. If the two gases are initially at the same pressure, at the time of impact there will be formed two shocks waves, one in each gas. If the initial pressures are unequal, there is the possibility of the shock wave in the high-pressure gas being replaced by a rarefaction. After the collision, the pressure and the velocity of the gases at the interface must be continuous. However, the density of the gases need not be the same on the two sides of the interface.

In case the initial pressures in the two gases are different, there is a critical ratio of initial pressures. If the difference in initial pressures is smaller than the critical value, there will still be two shock waves as in the case of equal initial pressures. However, if the difference in initial pressures is larger than the critical value, the shock wave in the high-pressure gas is replaced by a rarefaction. The high pressure gas expands at constant entropy and the Riemann method can be used to determine its conditions of flow.

[redacted]

The shock-wave method given here can be used to determine the conditions in the low-pressure gas. The problem is completely determined by the requirement that the velocity and pressure must be equal on both sides of the interface. This type of problem can be solved but it is usually impossible to obtain any analytical solution.

Let us consider the simpler problem of the two gases having the same initial pressure. It is convenient to specialize still further and let both gases be ideal and have the same value of γ. However, the gases may have different chemical compositions and densities. We suppose that before the collision, the pressure, velocity, and density is uniform in each of the two gases. The gas masses extend infinitely far in the plus and minus x directions respectively so as to avoid difficulties arising from end effects. To realize this experimentally it would be necessary to have the gases enclosed in long tubes with thin membranes at each end. These tubes would be thrown together and at the instant of impact the membranes would be removed. The mathematical treatment is much simpler.

Fig. 6 illustrates the conditions. The problem is to find the velocity of the shock waves, the velocity of the interface, and the pressure at the interface.

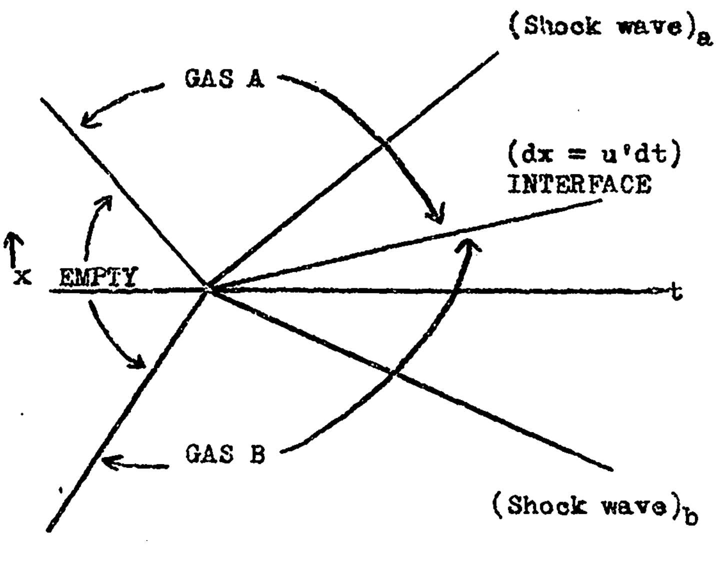

Figure 6

[redacted]

[redacted]

Let us designate conditions in gas A by the subscript a. Initially, the pressure, velocity, and density in gas A are p, u_{1a}, and ρ_{1a} respectively. The velocity of the shock wave is U_a. We can use the equations which we derived to consider the conditions on the two sides of the shock wave:

$$D_{1a} = U_{1a} - U_a \tag{74}$$

$$D_{2a} = U_{2a} - U_a \tag{75}$$

$$\xi_a = p_{2a}/p \tag{76}$$

$$\eta_a = \rho_{2a}/\rho_{1a} \tag{77}$$

But from Eqs. (52) and (56):[9]

$$D_{2a} = M_a/\rho_{2a} = -\sqrt{\frac{\rho_{1a}}{\rho_{2a}}\left(\frac{p_{2a} - p}{\rho_{2a} - \rho_{1a}}\right)} = -\sqrt{\frac{p}{\rho_{1a}}}\sqrt{\frac{\xi - 1}{\eta_a(\eta_a - 1)}} \tag{78}$$

$$D_{1a} = (\rho_{2a}/\rho_{1a})\, D_{2a} = \eta_a D_{2a} \tag{79}$$

So that

$$U_a = u_{1a} + \eta_a \sqrt{p/\rho_{1a}}\sqrt{\frac{\xi - 1}{\eta_a(\eta_a - 1)}} \tag{80}$$

$$u_{2a} = u_{1a} + \sqrt{p/\rho_{1a}}\sqrt{(1/\eta_a)(\eta_a - 1)(\xi - 1)} \tag{81}$$

And from Eq. (67)

$$\eta_a = \left[(\gamma - 1) + (\gamma + 1)\xi\right] \Big/ \left[(\gamma + 1) + (\gamma - 1)\xi\right] \tag{82}$$

Similarly if we designate conditions in gas B by the subscript b:

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9) Notice the use of the minus sign in the following equation. The necessity for it is clear from Fig. 6 since the material in crossing the shock wave, continues to travel in the negative x direction. [redacted]

[redacted]

$$D_{2b} = \sqrt{p(\xi_b - 1)/\rho_{1b}\,\eta_b(\eta_b - 1)} \tag{83}$$

$$U_b = u_{1b} - \eta_b \sqrt{p/\rho_{1b}} \sqrt{(\xi_b - 1)/\eta_b(\eta_b - 1)} \tag{84}$$

$$u_{2b} = u_{1b} - \sqrt{(p/\rho_{1b})} \sqrt{(1/\eta_b)\,(\eta_b - 1)\,(\xi_b - 1)} \tag{85}$$

$$\eta_b = \frac{(\gamma - 1) + (\gamma + 1)\xi_b}{(\gamma + 1) + (\gamma - 1)\xi_b} \tag{86}$$

Now we require that the material on both sides of the interface has the velocity u' and the pressure p'. The velocity of the interface is then also u'. We shall try to satisfy the equations with the pressure p' and the velocity u' for all of the material of both gases after the passage of the shock waves. Thus:

$$u' = u_{2a} = u_{2b} \tag{87}$$

$$\begin{aligned} p' &= p_{2a} = p_{2b} \\ &= p\xi_a = p\xi_b \end{aligned} \tag{88}$$

We must then solve the four equations: (82), (86), (87), (88), for the four unknowns ξ_a, η_a, ξ_b, η_b. From Eq. (88) it is clear that

$$\xi = \xi_a = \xi_b \tag{89}$$

Substituting this value of ξ into Eqs. (82) and (86):

$$\eta = \eta_a = \eta_b = \frac{(\gamma - 1) + (\gamma + 1)\xi}{(\gamma + 1) + (\gamma - 1)\xi} \tag{90}$$

[redacted]

And Eq(87) becomes:

$$u_{1a} + \sqrt{p/\rho_{1a}} \sqrt{(1/\eta)(\eta - 1)(\xi - 1)} = u_{1b} - \sqrt{(p/\rho_{1b})} \sqrt{(1/\eta)(\eta - 1)(\xi - 1)} \tag{91}$$

Rearranging Eq. (91) and squaring both sides of the equation:

$$(1 - 1/\eta)(\xi - 1) = \left(\frac{u_{1b} - u_{1a}}{\sqrt{p/\rho_{1a}} + \sqrt{p/\rho_{1b}}} \right)^2 \tag{92}$$

Or making use of Eq. (90) and letting

$$B = \left(\frac{u_{1b} - u_{1a}}{\sqrt{p/\rho_{1a}} + \sqrt{p/\rho_{1b}}} \right)^2 \tag{93}$$

$$2(\xi - 1)^2 = B\left[(\gamma - 1) + (\gamma + 1)\xi\right], \tag{94}$$

But Eq. (94) is a simple quadratic equation for ξ having the solution:

$$\xi = 1 + (B/4)(\gamma + 1) \pm \sqrt{B\gamma + (B^2/16)(\gamma + 1)^2} \tag{95}$$

Since B is always positive, both of the roots of Eq. (95) are real and therefore might correspond to solutions of the hydrodynamical equations. However, if we took the negative sign for the square root, ξ would be less than unity and the entropy would decrease when the material passed through the shock waves. This is obviously impossible so we must use the positive sign. All of the properties of the collision process are then completely determined.

[redacted]

[redacted]

III. TWO-DIMENSIONAL HYDRODYNAMICS

Lectures by von Neumann and Peierls
and Report by Fuchs

(9). STATIONARY TWO-AND THREE-DIMENSIONAL FLOWS. AVORTICITY. BERNOULLI's EQUATION.

The problem of two-and three-dimensional flows are considerably more difficult than the corresponding one-dimensional ones. Two- and three-dimensional problems must be treated by special methods which are applicable only to a limited class of problems.

In vector notation, the equation of motion is

$$\partial \underline{u}/\partial t + \underline{u} \circ \nabla \underline{u} = -(1/\rho)\nabla p \tag{96}$$

and the equation of continuity is

$$\partial \rho/\partial t = -\nabla \circ (\rho \underline{u}) \tag{97}$$

Here $\underline{u}$ is the velocity.

Flow problems naturally divide themselves into two classes: those involving vortices and those which are irrotational. Only a limited number of vortex problems can be solved and we will not consider them here. Whenever a fluid has a vortex, its angular velocity, $\underline{\omega}$, is different from zero in some region. Since $\underline{\omega} = (1/2)\nabla \times \underline{u}$, the requirement that there be no vortices is equivalent to requiring the curl of the velocity to be zero. But any vector whose curl is zero is the gradient of a scalar. Therefore we can set

$$\underline{u} = -\nabla \Phi \qquad \text{IRROTATIONAL FLOW} \tag{98}$$

Here Φ is called the velocity potential.

If a flow is initially vortex-free it will remain vortex-free if the pressure is a function of the density alone. Therefore there will be no

[redacted]

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[redacted]

vorticity generated as long as the flow remains isentropic. Nonconstant shocks will change entropy in a nonuniform manner and vortices can be formed. A plane shock is essentially a problem in one-dimensional flows and does not produce vortices. However, Hadamard showed that any shocks except those having either plane or spherical symmetry produce vortices. For example, vortices are formed when two plane shocks collide at an angle.

The three scalar equations corresponding to the vector equation of motion can be reduced to one equation when we make use of the velocity potential for irrotational flows. Eq. (96) becomes:

$$-(\partial/\partial t)\,(\nabla\Phi) + \nabla(\underline{u}\cdot\underline{u}/2) = -(1/\rho)\nabla p \qquad (99)$$

or inverting the order of differentiation for the first term:

$$-\nabla(\partial\Phi/\partial t) + \nabla(\mathrm{grad}\ \Phi)^2/2 = -(1/\rho)\nabla p \qquad (100)$$

This is equivalent to the equation

$$-d(\partial\Phi/\partial t) + d(\mathrm{grad}\ \Phi)^2/2 + (1/\rho)dp = 0 \qquad (101)$$

And integrating along any path

$$\int dp/\rho = \partial\Phi/\partial t - (1/2)\,(\mathrm{grad}\ \Phi)^2 + (1/2)\Pi \qquad (102)$$

Here Π is the constant of integration. For a steady state $\partial\Phi/\partial t = 0$ and we get Bernoulli's theorem:

$$\int dp/\rho + (1/2)\,(\mathrm{grad}\ \Phi)^2 = (1/2)\Pi \qquad (103)$$

Here Π is a constant which can be determined by knowing the velocity and pressure at some point in the fluid. Usually Π is evaluated from the pressure at a stagnation point where the fluid velocity is zero.

[redacted]

[redacted]

For an ideal gas, we can evaluate $\int dp/\rho$ and obtain an upper limit to the velocity of the flow:

$$p = k(S_o)\rho^{\gamma} \qquad (104)$$

$$\int dp/\rho = \left[\gamma k(S_o)^{1/\gamma}/(\gamma-1)\right] \; \rho^{(\gamma-1)/\gamma} = \left[\gamma k(S_o)/(\gamma-1)\right] \; \rho^{\gamma-1} \qquad (105)$$

But the velocity of sound, c, is given by

$$c = \sqrt{(\partial p/\partial \rho)_{S_o}} = \sqrt{\gamma k(S_o)\rho^{\gamma-1}} \qquad (106)$$

So that:

$$\int dp/\rho = c^2/(\gamma-1) \qquad (107)$$

And Bernoulli's equation becomes:

$$(\text{Velocity})^2 = (\text{grad } \Phi)^2 = \left[2/(\gamma-1)\right] \left[c_o^2 - c^2\right] \qquad (108)$$

Here c_o is the velocity of sound at a point where the flow velocity is zero.

The equation of continuity is also important. If we introduce the velocity potential into Eq. (97):

$$\partial\rho/\partial t = \nabla\cdot(\rho\nabla\Phi) = \rho\nabla\cdot\nabla\Phi + \nabla\rho\cdot\nabla\Phi \qquad (109)$$

For a steady state, $\partial\rho/\partial t = 0$ and

$$\nabla\cdot\nabla\Phi + \nabla \ln\rho\cdot\nabla\Phi = 0 \qquad (110)$$

Sometimes, the velocity potential may be determined in the following manner. The density is eliminated from Eq. (110) by making use of the implicit dependency of $\int dp/\rho$ on density in Eq.(103). This leads to a single differential equation involving Φ alone. The equation is very complicated and

[redacted]

[redacted]

nonlinear. When the effect of compressibility is small, it is possible to use this equation to obtain good approximations for the velocity potential.

(10) STATIONARY TWO-DIMENSIONAL FLOWS: NOZZLE FLOWS.

The steady-state flow through a nozzle provides one of the simplest examples of the use of the Bernoulli theorem. Consider a well-tapered (Laval) nozzle[10] attached to a large chamber with a large cross-sectional area. Suppose that the gas in the chamber is at rest, then the velocity, u, of the gas at any point is given by the Bernoulli equation (108):

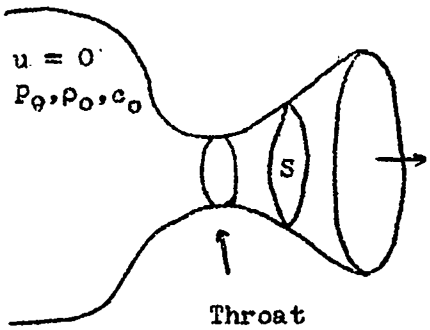

Fig. 7

$$u^2 = \left[2/(\gamma-1)\right] \left[c_o^2 - c^2\right] \qquad (108)$$

But $$c^2 = \gamma k(S_o)\rho^{\gamma-1} \qquad (106)$$

And $$p_o = k(S_o)\rho_o^{\gamma} \qquad (104)$$

So that

$$c^2 = (\gamma p_o/\rho_o)\ (p/p_o)^{(\gamma-1)/\gamma} \qquad (111)$$

Therefore Eq. (108) becomes:

$$u^2 = \left[2\gamma/(\gamma-1)\right] (p_o/\rho_o) \left[1 - (p/p_o)^{(\gamma-1)/\gamma}\right] \qquad (112)$$

However the rate of mass flow through any cross section S of the nozzle, M, must be the same at any point in the nozzle. Thus:

$$M = S\rho u = \text{constant} \qquad (113)$$

So combining Eq. (112) for the velocity with Eq. (104) for the adiabat:

$$M = S\rho_o\ (p/p_o)^{1/\gamma} \sqrt{\left[2\gamma/(\gamma-1)\right]\ p_o/\rho_o}\ \sqrt{1 - (p/p_o)^{(\gamma-1)/\gamma}} \qquad (114)$$

10) One of the best references for nozzle flows is Stadola and Lowenstein, "Steam and Gas Turbines" (McGraw-Hill, 1927) Vol. I.

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or rearranging:

$$S = \left[M \Big/ \sqrt{[2\gamma/(\gamma+1)]\; p_o \rho_o} \right] (p/p_o)^{-1/\gamma} \left[1 - (p/p_o)^{(\gamma-1)/\gamma} \right]^{-1/2} \qquad (115)$$

This equation gives the cross-sectional area as a function of the expansion ratio.

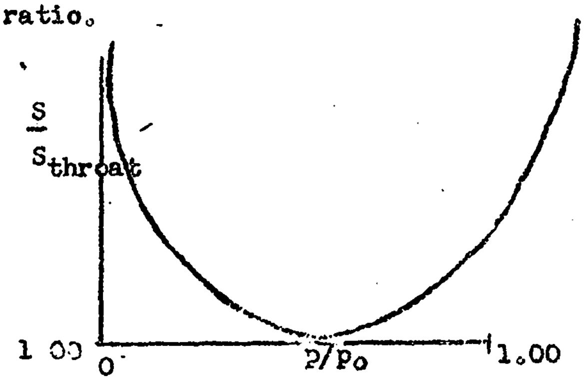

Figure 8

The conditions at the throat are particularly interesting. Here the cross-sectional area as a function of p/p_o passes through a minimum. In order to find the conditions at the minumum, let $y = (p/p_o)^{1/\gamma}$.

Then

$$S = (\text{constant}) \left[y^2 - y^{\gamma+1} \right]^{-1/2} \qquad (116)$$

At the throat (use subscript t to designate throat):

$$\frac{dS}{dy} = 0 = \frac{-(1/2)(\text{constant})}{\left[y_t^2 - y_t^{\gamma+1} \right]^{3/2}} \left[2y_t - (\gamma + 1) y_t^{\gamma} \right] \qquad (117)$$

Therefore:

$$y_t = \left[2/(\gamma+1) \right]^{1/(\gamma-1)} = (p_t/p_o)^{1/\gamma} \qquad (118)$$

Hence:

$$p_t/p_o = \left[2/(\gamma+1) \right]^{\gamma/(\gamma-1)} \qquad (119)$$

Substituting this ratio into Eq. (108), the velocity at the throat is given by the relation:

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$$u_t = \sqrt{\left[2\gamma/(\gamma+1)\right] p_o/\rho_o} \qquad (120)$$

The density at the throat is given by the adiabat:

$$\rho_t/\rho_o = \left[2/(\gamma+1)\right]^{1/(\gamma-1)} \qquad (121)$$

Using the perfect-gas equation, the temperature at the throat is given by:

$$T_t/T_o = 2/(\gamma+1) \qquad (122)$$

And of course

$$M = S_t\rho_t u_t = S_t\rho_o \left[2/(\gamma+1)\right]^{1/(\gamma-1)} \sqrt{\left[2\gamma/(\gamma+1)\right] p_o/\rho_o} \qquad (123)$$

The above equations apply until the gas has overexpanded so that the pressure in the nozzle is less than the external pressure. Under these conditions, plane shock waves may be expected (see Frank J. Malina, J. Franklin Inst. 230, 433 (1940)).

If the nozzle has too large an angle of taper (usually over 30°), the gases do not completely fill the nozzle and therefore do not expand as rapidly as might otherwise be expected.

If the nozzle is not tapered sufficiently in the neighborhood of the throat, the effect of the turbulent boundary layer becomes important. Von Karman has shown both theoretically and experimentally that under such conditions the boundary layer varies periodically along the nozzle and gives effectively a succession of constrictions, and the gas suffers a series of plane shocks in passing through this region.

The problem of gas flow through a straight tube is exceedingly complicated. In this problem the flow is determined by the friction due to turbulence in the boundary layer along the surface. In passing through the tube, the gas periodically overexpands, suffers a shock wave, and expands again. (see W. Frössel, N.A.C.A. Technical Memorandum No. 844).

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(11). STATIONARY TWO-DIMENSIONAL FLOWS: CORNER FLOWS. OBLIQUE SHOCKS. HEADWAVES OF WEDGES. INTERPRETATION. PITOT TUBE.

Usually when a two-dimensional flow traveling with supersonic velocity collides obliquely with an obstacle, it forms an oblique shock wave. This is evident in the photographs of bullets in flight (G. I. Taylor has an excellent article on this subject in Durand's Aerodynamics, Vol. III, p 236). The shock wave is a plane discontinuity with the material flowing through it obliquely. In discussing the flow, it is convenient to consider the shock wave as fixed and the gas moving obliquely through it. By superimposing on the whole system a velocity parallel to the shock wave, the problem can be reduced to the one-dimensional flow through a fixed shock wave. This type of shock wave cannot occur when everything is continuous. For example, these shock waves are formed on the sharp point of the bullet nose.

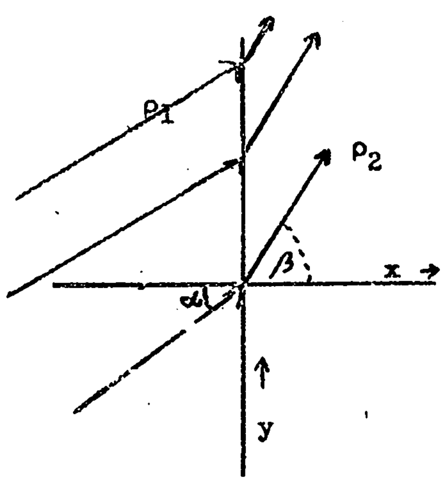

Figure 9

Figure 9 illustrates the problem of the oblique shock. The fluid hits the oblique shock at the angle α and departs with the angle β. We assume that the pressure is only a function of x.[11)] This has as a result that v, the component of velocity in the y direction, is constant. As usual, we consider only the problem of the steady state. In this case, Bernoulli's theorem is valid for the flows both before and after the oblique shock. If we let u be the component of velocity in the x direction, w be the total velocity and reserve the subscripts 1 and 2 for the flow before and after the shock wave.

11) The equation of motion for v is $\partial v/\partial t + u\, \partial v/\partial x = 0$. But for a steady state, $\partial v/\partial t = 0$ so that $\partial v/\partial x = 0$ and v is a constant.

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$$u_1 = W_1 \cos\alpha \qquad u_2 = W_2 \cos\beta$$

$$v_1 = W_1 \sin\alpha \qquad v_2 = W_2 \sin\beta \tag{124}$$

But

$$v_1 = v_2 \tag{125}$$

And from the conservation of matter,

$$M = \rho_1 u_1 = \rho_2 u_2 \tag{126}$$

So that

$$\tan\beta = v_2/u_2 = (v_1/u_1)\ \rho_2/\rho_1 = (\rho_2/\rho_1)\ \tan\alpha \ = \eta \tan\alpha \tag{127}$$

The rest of the solution proceeds much as in the case of the plane shocks.

$$M = \rho_1 u_1 = \sqrt{\left[(p_2 - p_1)/(\rho_2 - \rho_1)\right]\ \rho_1 \rho_2} \tag{128}$$

So that

$$u_1^2 = (\rho_2/\rho_1)\ \left[(p_2 - p_1)/(\rho_2 - \rho_1)\right] = (p_1/\rho_1)\ \left[\eta(\xi - 1)/(\eta - 1)\right] \tag{129}$$

But according to the perfect gas adiabat

$$c^2 = (\partial p/\partial \rho)_{S_0} \ = \gamma p/\rho \tag{130}$$

And therefore (making use of Eq. (124)):

$$u_1^2/c_1^2 = (\eta/\gamma)\ (\xi - 1)/(\eta - 1) = (W_1^2/c_1^2)\ \cos^2\alpha \tag{131}$$

Just as in the case of the normal plane shock,

$$\eta = \left[(\gamma - 1) + (\gamma + 1)\xi\right] \Big/ \left[(\gamma + 1) + (\gamma - 1)\xi\right] \tag{132}$$

Or solving Eq. (132) for ξ

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$$\xi = \left[\eta(\gamma + 1) - (\gamma - 1)\right] \Big/ \left[(\gamma + 1) - \eta(\gamma - 1)\right] \qquad (133)$$

Substituting this expression for ξ into Eq. (131)

$$(W_1/c_1)^2 \cos^2\alpha = 2\eta \left[(\gamma + 1) - (\gamma - 1)\eta\right]^{-1} \qquad (134)$$

And solving this equation for η

$$\eta = \frac{(\gamma + 1)\,(W_1/c_1)^2 \cos^2\alpha}{2 + (\gamma - 1)\,(W_1/c_1)^2 \cos^2\alpha} \qquad (135)$$

As in the case of the normal plane shocks, the fluid must flow across the shock in such a direction that ρ_2 is greater than ρ_1; also, the increase in density remains finite no matter how strong the shock. For a very strong shock, $\eta = (\gamma + 1)/(\gamma - 1)$. Thus making use of Eq. (127) and $\gamma = 1.4$, we find the following angles of deflection for a strong shock in air:

α	β	$\beta - \alpha$
0°	0°	0°
15	58	43
30	74	44
45	80	35
60	84	24
75	87	12
90	90	0

It is clear from the above table that the deflection, $\beta - \alpha$, cannot exceed some maximum value. If the deflection is greater than this, the disturbance cannot produce a stationary shock. We can find this maximum angle of deflection for a given value of the initial velocity, W_1, and initial velocity of sound, c_1, in the following manner: differentiating both sides of Eq. (127) with respect to α

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$$\sec^2 \beta \; d\beta/d\alpha = \tan \alpha \; d\eta/d\alpha + \eta \sec^2 \alpha \qquad (136)$$

Subtracting $\sec^2\beta$ from both sides of this equation

$$\sec^2\beta \; d(\beta - \alpha)/d\alpha = \tan \alpha \; d\eta/d\alpha + \eta \sec^2\alpha - \sec^2\beta \qquad (137)$$

But the condition that $\beta - \alpha$ should be a maximum or minimum is for $d(\beta - \alpha)/d\alpha = 0$. Making use once more of Eq. (127) to eliminate β, the condition for $\beta - \alpha$ being a maximum becomes:

$$0 = \tan \alpha \; d\eta/d\alpha + \eta \sec^2\alpha - 1 - \eta \tan^2\alpha \qquad (138)$$

And differentiating η in Eq. (135) with respect to α keeping W_1 and c_1 constant:

$$d\eta/d\alpha = \tan \alpha \left[-2\eta + 2\eta^2 \; (\gamma - 1)/(\gamma + 1) \right] \qquad (139)$$

Substituting Eq. (139) into Eq. (138) and solving for $\tan^2\alpha$, the condition for the maximum deflection becomes:

$$\tan^2\alpha = \frac{\eta - 1}{\eta + (3 - \gamma)/(\gamma + 1)\,\eta^2} \qquad (140)$$

Thus for air with $\gamma = 1.4$, we get the following conditions for maximum deflection:

η	$\tan^2\alpha$	$(W_1/c_1)^2$	$\beta - \alpha$
1	0	1	0
2	.273	3.18	$19^\circ 40'$
3	.222	6.11	$29^\circ 30'$
4	.204	12.05	$37^\circ 50'$
5	.184	29.6	$41^\circ 50'$
6	.167		46°

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For air, $\eta = 6$ is the greatest compression which can be obtained with the strongest shocks.

The above treatment was first made by Prandtl and Mayer. This analysis may be applied directly to the problem of gases flowing past an infinite wedge. After the shock, the gas moves parallel to the surface of the wedge. Therefore the deflection, $\beta - \alpha$, is equal to the half angle of the wedge, θ_s. This is shown in Figure 10.

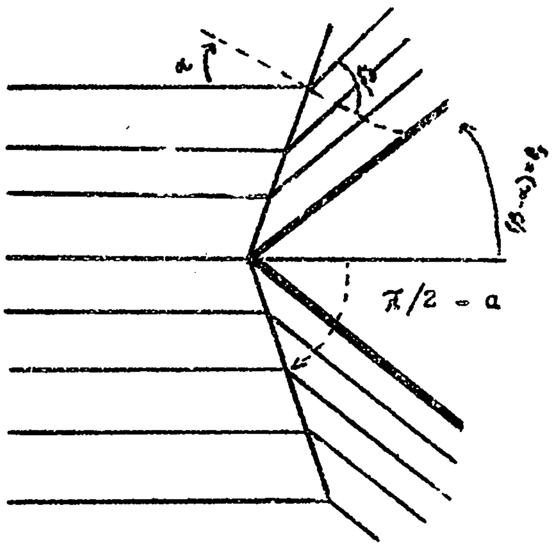

Figure 10

On a photograph, such as of a bullet in flight, the angle, $\pi/2 - \alpha$, that the shock makes with the wedge is clearly visible. Knowing both $\beta - \alpha$ and α, we can get the compression ratio, η, from Eq. (127) and then the pressure increase, ξ, from Eq. (133). Knowing the incident pressure and density, we get c_1 from Eq. (130). Thus we can obtain the velocity, w_1, of the gas with respect to the wedge from Eq. (134). This then comprises a complete solution.

If the wedge is infinite in extent, the oblique shock waves remain attached to the point of the wedge until the maximum value of $\beta - \alpha$ is reached. However, von Neumann has shown that if the wedge is finite, the shock waves detach themselves from the point of the wedge when $w_2/c_2 = 1$. This condition is reached for wedges with half angles one or two degrees less than the maximum value of $\beta - \alpha$. Figure 11 shows this situation. Here the solution is stationary and the headwave remains a finite distance in front of the wedge, the smaller is the breadth of the wedge, the farther the headwave remains away from the wedge. We can find the angle for which $w_2 = c_2$ in the following manner.

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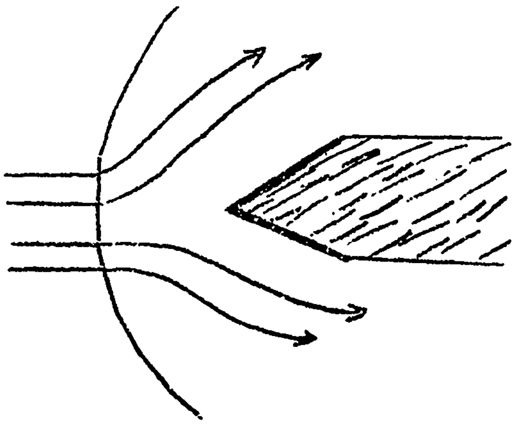

Figure 11.

Making use of Eqs. (124), (125), and (126):

$$W_2^2 = u_2^2 + v_2^2 = (\rho_1/\rho_2)^2\ u_1^2 + v_1^2$$
$$= (1/\eta^2 + \tan^2\alpha)\ W_1^2 \cos^2\alpha \qquad (141)$$

But from Eq. (130),

$$c_2^2 = c_1^2\ (p_2^2/p_1^2)\ \ \rho_1^2/\rho_2^2 = c_1^2\ \xi^2/\eta^2 \qquad (142)$$

So that using Eq. (134) and (133)

$$W_2^2/c_2^2 = \left[(1 + \eta^2\ \tan^2\alpha)/\xi^2\right]\ (W_1^2/c_1^2)\cos^2\alpha \qquad (143)$$
$$= \frac{(1 + \eta^2\tan^2\alpha)2\eta\left[(\gamma+1) - (\gamma-1)\eta\right]}{\left[\eta(\gamma+1) - (\gamma-1)\right]^2}$$

Therefore the condition that $W_2 = c_2$ becomes:

$$\tan^2\alpha = \frac{(\gamma - 1)^2 - 2\eta\gamma(\gamma + 1) + \eta^2(\gamma^2 + 4\gamma - 1)}{2\eta^3\ \left[(\gamma + 1) - (\gamma - 1)\eta\right]} \qquad (144)$$

If the velocity past the shock wave, w_2 , is less than the velocity of sound in this region, c_2 , the disturbance at the far corners of the wedge travels back towards the point of the wedge and affects the shock wave, causing the detachment. For a finite wedge, the situation indicated in Figure 11 holds for angles larger than the critical angles. For an infinite wedge having a half angle greater than the maximum $\beta - \alpha$, the head wave becomes detached and travels back through the fluid. This gives rise to a nonstationary solution.

The problem of headwaves for conical wedges (projectiles) has been treated by Taylor and Maccoll, etc. The phenomena are similar to those for

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wedges but the analysis is considerably more difficult because the conditions of pressure, density, and velocity downstream can no longer be constant and satisfy the equation of continuity.

Pitot tubes are designed to measure the velocity of a gas flow in terms of pressure. Effectively they form a wedge with α and β equal to zero. A tube extends from the gas stream to the pressure gage. The tube is constructed so that the gas velocity at the pressure gage is effectively zero. The conditions at the pressure gage (which we shall designate by the subscript 3) are related to the conditions just in back of the shock wave by the Bernoulli equation:

$$(1/2)\, u_2^2 = \left[\gamma/(\gamma-1)\right]\, (p_3/\rho_3)\, \left[1 - (p_2/p_3)^{(\gamma-1)/\gamma}\right] \qquad (145)$$

But the conditions at the points 2 and 3 satisfy the same adiabat so that

$$\rho_3 = (p_3/p_2)^{1/\gamma}\rho_2 \qquad (146)$$

Substituting this into Eq. (145) and rearranging:

$$(p_3/p_2)^{(\gamma-1)/\gamma} = 1 + \left[(\gamma-1)/2\gamma\right]\, u_2^2\rho_2/p_2 \qquad (147)$$

But according to Eqs. (126) and (129):

$$u_2^2 = u_1^2(\rho_1/\rho_2)^2 = (p_1\rho_1/\rho_2^2)\, \left[\eta(\xi-1)/(\eta-1)\right] = (p_1/\rho_2)\, (\xi-1)/(\eta-1) \qquad (148)$$

So that Eq. (147) becomes:

$$(p_3/p_2)^{(\gamma-1)/\gamma} = 1 + \left[(\gamma-1)/2\gamma\right] (\xi-1)/\xi(\eta-1) \qquad (149)$$

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And substituting the expression for ξ from Eqs. (152) and rearranging:

$$(p_3/p_2)^{(\gamma-1)/\gamma} = (\gamma+1)^2/4\gamma + \left[(\gamma^2-1)/4\gamma\right] 1/\xi \qquad (150)$$

After multiplying both sides of the equation by

$$\xi^{(\gamma-1)/\gamma} = (p_2/p_1)^{(\gamma-1)/\gamma}$$

and taking the $\gamma/(\gamma-1)$ root of both sides of the equation:

$$p_3/p_1 = \xi \left[(\gamma+1)^2/4\gamma + (\gamma^2-1/4\gamma)\ 1/\xi\right]^{\gamma/(\gamma-1)} \qquad (151)$$

Where ξ is obtained from Eqs. (133) and (135) after setting $\alpha = 0$:

$$\xi = \left[2\gamma/(\gamma+1)\right] (W_1/c_1)^2 - (\gamma-1)/(\gamma+1) \qquad (152)$$

Thus the pitot tube measurement of p_3 determines the gas velocity, w_1 , if the initial pressure and density of the gas are known.

(12) STATIONARY TWO-DIMENSIONAL FLOWS: FLOW AROUND CONVEX CORNER(RAREFACTION)

The supersonic flow of a gas around a convex corner leads to a rarefaction instead of a shock. Figure 12 shows the flow. The gas maintains constant pressure, density and velocity until it reaches the line OO' where the disturbance from the corner first reaches it. The streamlines turn almost radially about one corner and then become parallel to the new surface.

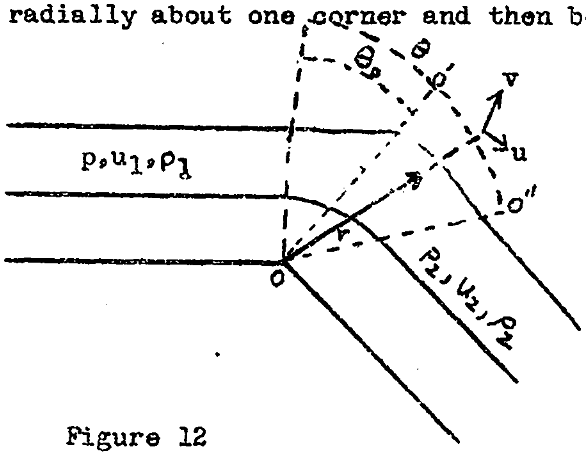

Figure 12

Originally the pressure, density, and the velocity are uniform and they become uniform again after passing through the rarefaction.

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It is convenient to use polar coordinates with the corner as center for this treatment and we shall let u be the component of velocity perpendicular to the radius vector and v be the velocity in the direction of increasing radius vector.

The line OO' is at an angle, $\theta_0 = \pi/2 - \psi$, where ψ is the Mach angle,

$$\sin \psi = c_1/u_1$$

The meaning of the Mach angle is clear from Figure 13. The rarefaction disturbance at O travels with the velocity c_1, it is swept downstream with the velocity of the fluid, u_1. Therefore, the farthest upstream it can reach is along the line OO'.

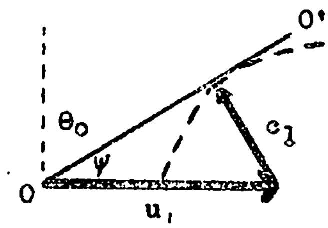

Figure 13

In rectangular coordinates, the equations of motion and the equation of continuity for a stationary flow may be written:

$$u_x \;\partial u_x/\partial x + u_y \;\partial u_x/\partial y = -(1/\rho)\;\partial p/\partial x \qquad (153)$$

$$u_x \;\partial u_y/\partial x + u_y \;\partial u_y/\partial y = -(1/\rho)\;\partial p/\partial y \qquad (154)$$

$$\partial(u_x\rho)/\partial x + \partial(u_y\rho)/\partial y = 0 \qquad (155)$$

Here we have let u_x and u_y be the velocity in the x and y directions respectively. In order to express these equations in polar coordinates, it is necessary to set:

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$$u_x = u \cos\theta + v \sin\theta \tag{156}$$

$$u_y = -u \sin\theta + v \cos\theta \tag{157}$$

$$\partial/\partial x = \sin\theta \ \partial/\partial r + (1/r)\cos\theta \ \partial/\partial\theta \tag{158}$$

$$\partial/\partial y = \cos\theta \ \partial/\partial r - (1/r)\sin\theta \ \partial/\partial\theta \tag{159}$$

From these relations, we obtain:

$$u \ \partial u/\partial r + (v/r) \ \partial u/\partial\theta - v^2/r = -(1/\rho) \ \partial p/\partial r \tag{160}$$

$$u \ \partial v/\partial r + (v/r) \ \partial v/\partial\theta + uv/r = -(1/\rho r) \ \partial p/\partial\theta \tag{161}$$

$$\partial(\rho u r)/\partial r + \partial(\rho v)/\partial\theta = 0 \tag{162}$$

These equations may be greatly simplified by the assumption that the velocity, pressure, and density in the rarefaction region are functions of θ but independent or r. The only necessary justification for this assumption is that we can satisfy all of the equations and obtain a formal solution of this type. The equation of motion and the equation of continuity then become:

$$(v/r) \ \partial u/\partial\theta - v^2/r = 0 \tag{160'}$$

$$(v/r) \ \partial v/\partial\theta + uv/r = -(1/\rho r) \ \partial p/\partial\theta \tag{161'}$$

$$\rho u + \partial(\rho v)/\partial\theta = 0 \tag{162'}$$

Furthermore:

$$\partial p/\partial\theta = (\partial p/\partial\rho) \ \partial\rho/\partial\theta = c^2 \ \partial\rho/\partial\theta \tag{163}$$

The Eqs. (160'), (161'), and (162') therefore reduce to:

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$$\partial u/\partial\theta = v \qquad (160'')$$

$$v\left[\partial v/\partial\theta + u\right] = -(c^2/\rho)\ \partial p/\partial\theta \qquad (161'')$$

$$v\left[u + \partial v/\partial\theta\right] = -(v^2/\rho)\ \partial\rho/\partial\theta \qquad (162'')$$

In order for Eqs. (161'') and (162'') to be compatible, $v = \pm c$ or $\partial\rho/\partial\theta = 0$ or $\partial v/\partial\theta + u = 0$. If $\partial v/\partial\theta + u = 0$, according to Eq. (161') it would follow that $\partial p/\partial\theta = 0$ and the pressure would everywhere be the same. Similarly if $\partial\rho/\partial\theta = 0$, the density is everywhere the same. Neither of these cases could be generally applicable. Therefore we conclude that

$$v = -c = -\sqrt{\gamma p/\rho} = -\sqrt{\gamma k(S_o)}\ \rho^{(\gamma-1)/2} \qquad (164)$$

From this it follows that

$$(1/\rho)\ \partial\rho/\partial\theta = \left[2/(\gamma-1)\right]\ (1/v)\ \partial v/\partial\theta \qquad (165)$$

And Eq. (162'') becomes:

$$u + \partial v/\partial\theta = -\left[2/(\gamma-1)\right]\ \partial v/\partial\theta \qquad (166)$$

Taking the derivative of both sides of this equation with respect to θ and making use of Eq. (160''):

$$\left[(\gamma+1)/(\gamma-1)\right]\ \partial^2 v/\partial\theta^2 = -\partial u/\partial\theta = -v \qquad (167)$$

The solution to this equation is

$$\left.\begin{aligned} v &= -A\sqrt{(\gamma-1)/(\gamma+1)}\ \sin\sqrt{(\gamma-1)/(\gamma+1)}\ (\theta + \delta) \\ u &= +A\ \cos\sqrt{(\gamma-1)/(\gamma+1)}\ (\theta + \delta) \end{aligned}\right\} \qquad (168)$$

Here A and δ are constants of integration to be determined in the following

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manner. According to Bernoulli's equation (108), the square of the velocity is given by:

$$u^2 + v^2 = \left[2/(\gamma-1)\right] \left[c_o^2 - c^2\right] \tag{169}$$

In the original flow, the constant c_o is determined by the relation:

$$u_1^2 = \left[2/(\gamma-1)\right] \left[c_o^2 - c_1^2\right] \tag{170}$$

or

$$\left[2/(\gamma-1)\right] c_o^2 = u_1^2 + \left[2/(\gamma-1)\right] c_1^2 \tag{171}$$

And since $v = -c$, Eq. (169) becomes

$$u^2 + \left[(\gamma+1)/(\gamma-1)\right] v^2 = 2c_o^2/(\gamma-1) \tag{172}$$

Substituting u and v from Eq. (168), we get

$$A^2 = 2c_o^2/(\gamma-1) = u_1^2 + 2c_1^2/(\gamma-1) \tag{173}$$

To evaluate δ , we set $v = -c_1$ when $\theta = \theta_o$

$$c_1 = + \sqrt{\left[(\gamma-1)/(\gamma+1)\right] \left[u_1^2 + 2c_1^2/(\gamma-1)\right]} \sin \sqrt{(\gamma-1)/(\gamma+1)}\,(\theta_o + \delta) \tag{174}$$

The change of the pressure with angle may be determined in the following manner. From the adiabat and $v = - c$ we obtain:

$$v^2 = c^2 = c_1^2 \,(p/p_1)^{(\gamma-1)/\gamma} \tag{175}$$

So that

$$v/(v)_{\theta=\theta_o} = - v/c_1 = (p/p_1)^{(\gamma-1)/2\gamma} =$$

$$\left[\sin \sqrt{(\gamma-1)/(\gamma+1)}\,(\theta + \delta)\right] \Big/ \sin \sqrt{(\gamma-1)/(\gamma+1)}\,(\theta_o + \delta) \tag{176}$$

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For very high velocities or sharp angles, the fluid flow cannot follow the contour of the corner and the flow forms a free surface. The slope of the velocity must change gradually. This slope is given by the expression:

$$\frac{u_y}{u_x} = \frac{-u \sin\theta + v \cos\theta}{u \cos\theta + v \sin\theta} =$$

$$\frac{-\sin\theta \cos\sqrt{(\gamma-1)/(\gamma+1)}\,(\theta+\delta) - \sqrt{(\gamma-1)/(\gamma+1)}\,\cos\theta \sin\sqrt{(\gamma-1)/(\gamma+1)}\,(\theta+\delta)}{+\cos\theta \cos\sqrt{(\gamma-1)/(\gamma+1)}\,(\theta+\delta) - \sqrt{(\gamma-1)/(\gamma+1)}\,\sin\theta \cos\sqrt{(\gamma-1)/(\gamma+1)}\,(\theta+\delta)} \tag{177}$$

If the corner goes from a surface which initially has the slope zero to a surface whose slope is $dy/dx = -m$, then θ changes from θ_0 to the value given by Eq. (177) if u_y/u_x is set equal to $-m$. After this point, the pressure, velocity, and density remains constant.

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(13). REFLECTION OF SHOCK WAVES FROM A RIGID WALL

The reflection of shock waves from a rigid wall and the collision between shock waves are important phenomena which lend themselves to direct experimental observations. For example the reflection of shock waves from a rigid wall is often used to measure the velocity or pressure of shock waves. Figure 14 shows such an experimental setup. The angle of the reflected shock wave tells the velocity of the blast wave if the pressure behind the blast is known or the pressure if the velocity is known. As the blast progresses it travels across the plate. The phenomenon is therefore not stationary with respect to space, but it may be stationary with respect to a co-ordinate system traveling with the blast wave.

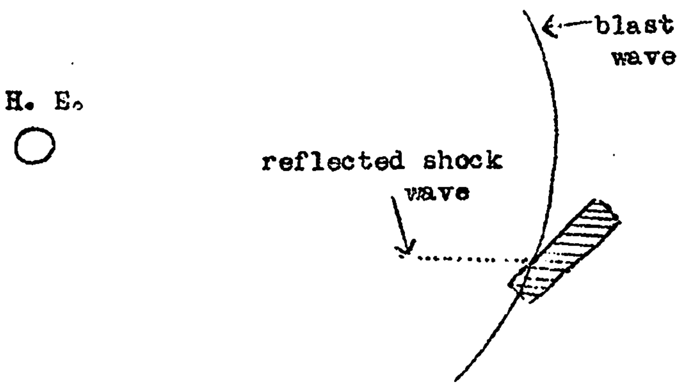

Figure 14

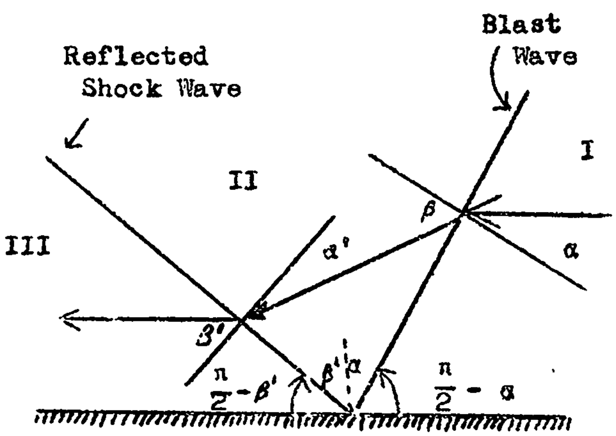

Figure 15

Suppose that the blast is sufficiently far away from the plate so that blast wave presents an essentially plane front. It strikes

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the plate at an angle $\pi/2 - \alpha$. If the blast is travelling at a velocity, U, the blast wave will travel along the plate with a velocity $U/\sin\alpha$. In the co-ordinate system fixed with respect to the blast wave, the material in the undisturbed region, I, has the velocity $-U/\sin\alpha$ parallel to the surface, and of course the initial pressure and density, p_1 and ρ_1. The fluid is deflected along straight lines towards the surface in region II. A reflected shock wave rectifies the flow and makes the fluid motion in region III once more parallel to the surface. Let us designate the results of passing through the original blast wave by unprimed letters and the results of passing through the reflected wave by primed letters; also, we use the subscripts 1, 2, and 3 to designate conditions in the three regions. Let us suppose that we know U, p_1, and ρ_1. Then the conditions in region II are completely determined. Setting $w_1 = U/\sin\alpha$, Eqs. (133) and (135) tell the pressure and density in region II. Eq. (127) tells the angle β. Eq. (143) gives the velocity of the fluid in region II. To get from region II to region III the angle of the reflected shock wave must be adjusted so that:

$$\beta' - \alpha' = \beta - \alpha \tag{178}$$

Here β and α are already known and β' and α' are connected by the relation:

$$\tan\beta' = \nu' \tan\alpha' \tag{179}$$

[redacted]

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and

$$\eta' = \frac{(\gamma + 1)(W_2/c_2)^2 \cos^2 \alpha'}{2 + (\gamma - 1)(W_2/c_2)^2 \cos^2 \alpha'} \qquad (180)$$

where

$$\left(\frac{W_2}{c_2}\right)^2 = \frac{(1 + \eta^2 \tan^2 \alpha)\, 2\eta\left[(\gamma + 1) - (\gamma - 1)\eta\right]}{\left[\eta(\gamma + 1) - (\gamma - 1)\right]^2} \qquad (14*)$$

As long as the shock is weak so that N is almost unity or as long as α is small, it is possible to find a value of α' which satisfies the requirement of Eq. (178). In the case of weak shocks, the reflected wave comes off at the acoustical angle, $\beta' = \alpha$. However, for larger angles or stronger shocks there is no solution of this nature and the problem is much more complicated.

For angles larger than the critical we have the picture shown in Figure 16. Next to the surface we have a Mach wave perpendicular to the surface and extending out a distance corresponding to a disturbance traveling with the Mach angle from the corner of the plate.* This distance therefore increases with time as the blast passes across the surface. Joined to the Mach wave is the original blast wave and the reflected wave. Behind the reflected shock wave is a small region of compression. The fluid which passes through the Mach wave has a higher temperature and a different density from the material which has passed through the two shock waves and therefore there is a slip stream separating the two gases (with no pressure gradiant across the slipstream).

* This is accurately the Mach angle only in a simple three-shock theory, and is observed to differ from it considerably.

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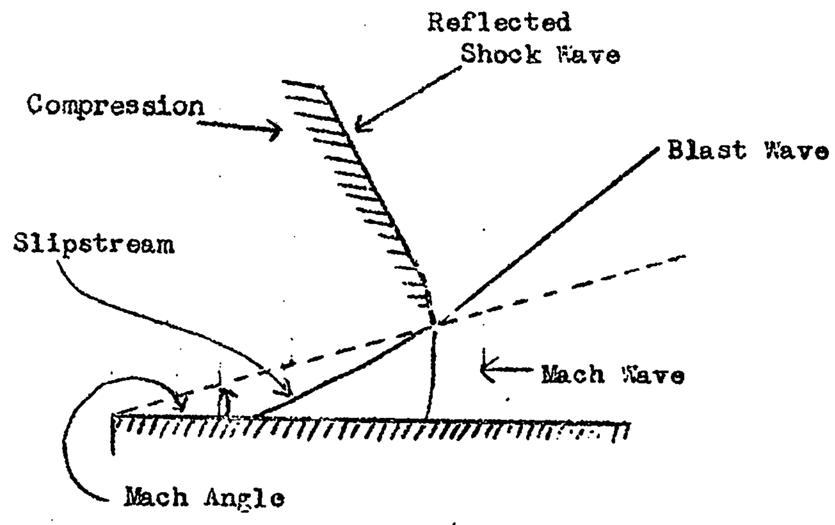

Figure 16

(14) TWO-DIMENSIONAL STATIONARY PROBLEMS - METHOD OF CHARACTERISTICS DEVELOPED BY FUCHS.

(This section has been written by K. Fuchs. Since the completion of this lecture series he has developed the following extension of the Riemann Method to two-dimensional problems. This makes it possible in principle to solve any problem involving stationary flows without vortices or shocks.)

For stationary flows, the equation of motion (96) and the equation of continuity (97) become:

$$\underline{u} \cdot \nabla \underline{u} + (1/\rho) \nabla p = 0 \qquad (181)$$

$$\underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} = 0 \qquad (182)$$

In addition we have an equation of state expressing the pressure in terms of

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the density and entropy. The entropy[12] remains constant along a streamline so that:

$$\underline{u} \cdot \nabla S = 0 \qquad (183)$$

Let us introduce a parameter ℓ which measures the length along a streamline and similarly a parameter n which measures the length along the normals to the streamlines. From the definition of ℓ :

$$\underline{u} \cdot \nabla = u \, d/d\ell \qquad u = |\underline{u}| \qquad (184)$$

If ϕ is the angle between the streamline and the x axis, then:

$$\tan \phi = u_y / u_x \qquad (185)$$

and

$$d/d\ell = \cos \phi \;\; \partial/\partial x + \sin \phi \;\; \partial/\partial y \qquad (186)$$

$$d/dn = - \sin \phi \;\; \partial/\partial x + \cos \phi \;\; \partial/\partial y \qquad (187)$$

We now transform the differential equations (181) through (183) into differential equations along the streamlines and their normals. Equation (183) has already the correct form since it is identical with

$$dS/d\ell = 0 \qquad (188)$$

A second equation is obtained from the equations of motion which yield the Bernoulli equation (similar to Eq. (101)):

$$\frac{d}{d\ell}\left\{\frac{1}{2} u^2 + \int \frac{dp}{\rho}\right\} = 0 \qquad (189)$$

Differentiating Eq. (185) along a streamline we find:

$$u \frac{d\phi}{d\ell} = \cos \phi \, \frac{du_y}{d\ell} - \sin \phi \, \frac{du_x}{d\ell} \qquad (190)$$

---

12) In everything that follows, any function of the entropy would work just as well as the entropy itself. For example, in a gas obeying the γ law, $pV^{\gamma} = k(S)$, it would be convenient to use k in place of entropy.

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And making use of Eqs. (181) and (184):

$$u^2 \frac{d\phi}{d\ell} = -\cos\phi \, \frac{1}{\rho} \, \frac{\partial p}{\partial y} + \sin\phi \, \frac{1}{\rho} \, \frac{\partial p}{\partial x} \qquad (191)$$

Then with Eqs. (186) and (187):

$$u^2\rho \, \frac{\partial\phi}{\partial\ell} + \frac{dp}{dn} = 0 \qquad (192)$$

The first term in this equation is the centrifugal force which is balanced by the second term corresponding to a pressure gradient normal to the streamline.

In order to express the equation of continuity in terms of ℓ and n, consider a fixed point, P_o. We can define a cartesian co-ordinate system with the origin at the point P_o and the x axis pointing in the direction of the streamline which passes through P_o. Then at P_o both u_y and du_y/dx vanish. Hence:

$$(\text{at } P_o) \qquad \frac{du}{d\ell} = \frac{\partial u_x}{\partial x} \qquad (193)$$

For a point, P, on a neighboring streamline we have to the first order:

$$(\text{at } P) \qquad \phi = \frac{d\phi}{dn} \, dn \qquad (194)$$

$$u_y = u\phi = \frac{\partial u_y}{\partial y} \, dn \qquad (195)$$

Hence

$$\frac{\partial u_y}{\partial y} = u \, \frac{d\phi}{dn} \qquad (196)$$

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Substituting Eqs. (184), (193), and (196) into the equation of continuity Eq. (182):

$$u \frac{d\rho}{d\ell} + \rho \frac{\partial u_x}{\partial x} + \rho \frac{\partial u_y}{\partial y} = u \frac{d\rho}{d\ell} + \rho \frac{du}{d\ell} + u\rho \frac{d\phi}{dn} = 0 \tag{197}$$

Since P_o was arbitrarily chosen, Equation (197) holds for any point.

We can now define two "characteristics" such that if a signal is emitted from any point, the disturbance created cannot reach farther upstream than the region bounded by the characteristics (see Figure 13, page 50; here 0 0' is a characteristic). The characteristics make the Mach angle, ψ, with the streamlines. Here

$$\sin\psi = c/u \tag{198}$$

There are + and − characteristics depending on whether the angle between the streamline and the characteristic is plus or minus ψ.

If we let $\lambda_{\pm}$ be the distance along a $\pm$ characteristic, then:

$$\frac{d}{d\lambda_{\pm}} = \cos\psi \frac{d}{d\ell} \pm \sin\psi \frac{d}{dn} \tag{199}$$

The equation of continuity (197) becomes (making use of Eqs. (191) and (199)):

$$\frac{d\phi}{d\lambda_{\pm}} = - \frac{\cos\psi}{u^2\rho} \frac{dp}{dn} \mp \frac{\sin\psi}{u\rho}\left[u \frac{d\rho}{d\ell} + \rho \frac{du}{d\ell}\right] \tag{200}$$

But the Bernoulli equation (189) can be written:

$$u \frac{du}{d\ell} + \frac{1}{\rho} \frac{dp}{d\ell} = 0 \tag{201}$$

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And we also have:

$$\frac{dp}{d\ell} = \left(\frac{dp}{d\rho}\right)_S \frac{d\rho}{d\ell} = c^2 \frac{d\rho}{d\ell} = \sin^2\psi\, u^2 \frac{d\rho}{d\ell} \tag{202}$$

Using these relations to eliminate $d\rho/d\ell$ and $du/d\ell$ from Eq. (200), we find:

$$\frac{d\phi}{d\lambda_\pm} = -\frac{\cos\psi}{u^2\rho}\frac{dp}{dn} \mp \frac{\cos^2\psi}{\sin\psi}\frac{1}{u^2\rho}\frac{dp}{d\ell} \tag{203}$$

Then making use of Eq. (199):

$$\frac{d\phi}{d\lambda_\pm} \pm \frac{\cot\psi}{\rho u^2}\frac{dp}{d\lambda_\pm} = 0 \tag{204}$$

We may use this equation together with Eqs. (188) and (189) which involve distance along the streamlines but not the distance normal to the streamlines. Or, alternatively, we may also eliminate the streamlines from the above equations by observing that:

$$\frac{d}{d\lambda_+} + \frac{d}{d\lambda_-} = 2\cos\psi\frac{d}{d\ell} \tag{205}$$

Hence from Eq. (188)

$$\frac{dS}{d\lambda_+} + \frac{dS}{d\lambda_-} = 0 \tag{206}$$

And therefore:

$$\left(\frac{d}{d\lambda_+} + \frac{d}{d\lambda_-}\right)\left(\frac{1}{2}u^2 + \int\frac{dp}{\rho}\right) = 0 \tag{207}$$

<u>Special Case</u> - No Vortices and Entropy Constant Throughout Fluid

Let us assume now that the motion is free of vortices and that the entropy is constant. These two conditions usually go together since both vorticity and varying entropy will in general be introduced into our problems by means of shocks of varying strength.

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The assumption of no vorticity leads to the identity:

$$\underline{u} \cdot \nabla \underline{u} = \frac{1}{2} \nabla u^2 \tag{208}$$

The assumption of constant entropy has as a consequence that p and ρ are functions of each other so that:

$$\frac{1}{\rho} \nabla p = \nabla \int \frac{dp}{\rho} \tag{209}$$

And the Bernoulli equation becomes (the same as Eq. (103)):

$$\frac{u^2}{2} + \int \frac{dp}{\rho} = \text{constant} = \Pi/2 \tag{210}$$

Here, in contrast to the more general case just considered, Π is a constant not only along one particular streamline but throughout all space.

Thus for a given value of Π, u is a unique function of the pressure. Since also c is a unique function of the pressure, the same is true of the Mach angle which is defined in terms of c and u. Hence we can define a function, F, by the integral:

$$F = \int \frac{\cot \psi}{\rho u^2} \, dp \tag{211}$$

For a given value of Π and a given value of the entropy, this function is a unique function of the pressure. If we wish to consider the density, ρ, as the independent variable rather than p, we may write Eq. (211) also in the form (by use of the relation $u^2 \sin^2\psi = c^2 = dp/d\rho$):

$$F = \int \frac{\sin \psi \cos \psi}{\rho} \, d\rho \tag{211'}$$

The equation (204) may be written in the form:

$$\frac{d}{d\lambda_\pm} (\phi \pm F) = 0 \tag{212}$$

Hence:

$$a_\pm = \phi \pm F \tag{213}$$

is constant along the corresponding characteristics.

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In general, each characteristic will have its own value of a_+. We obtain immediately the direction of the streamlines and the pressure at any point where two characteristics a_+ and a_- intersect. We need only write Eq. (213) in the form:

$$\phi = \frac{a_+ + a_-}{2} \tag{214}$$

$$F = \frac{a_+ - a_-}{2} \tag{215}$$

The material velocity, u, is then given by Eq. (210) and the Mach angle ψ is given by Eq. (198).

However, in order to find the position in space of the point of intersection of two characteristics, we have to integrate once more to obtain the equation of the characteristics which satisfies the differential equation:

$$dy/dx = \tan\,(\phi \pm \psi) \tag{216}$$

For uniform flow (i.e. ϕ and F are constant), it follows from Eqs. (214) and (215) that a_+ is the same for all + characteristics and a_- is the same for all - characteristics.

Consider now what happens when a region of uniform flow is joined by a region of nonuniform flow. This situation occurs when a fluid flowing with constant velocity along a plane wall comes to a bend in the wall. This is shown in Figure 17. If the bend starts at a point A, it will give rise to a disturbance affecting the region to the right of the + characteristic AB. The + characteristics in this region start from the wall. The - characteristics cross from the region of uniform flow and therefore a_- is the same for all - characteristics. Consider now a

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+ characteristic A'B' with a characteristic parameter a_+. Since a_- is constant, it follows from Eq. (214) and (215) that ϕ and F have the same value for all points. In other words, the direction of the streamlines, the pressure, and therefore also the material velocity and the Mach angles are constant along any + characteristic. Furthermore, since the direction, $\phi + \psi$, of the + characteristics is constant, the + characteristics are all straight lines. The − characteristics have of course all the same direction ($\phi - \psi$) when they cross a given + characteristic, but they change direction when crossing from one + characteristic to another.

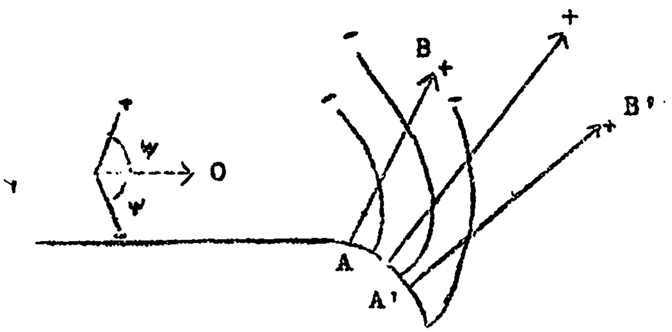

Figure 17

The quantity a_+ has to be determined from the condition that at the boundary, the direction of the streamline is in the direction of the wall. If ϕ_{wall} is the angle of the wall with the x axis at the point where the characteristic starts, and a_- is known from the properties of the flow in the region of uniform flow, then from Eq. (214):

$$a_+ = 2\,\phi_{wall} - a_- \tag{217}$$

and from Eq. (215):

$$F = \phi_{wall} + \frac{a_-}{2} \tag{218}$$

If the wall curves away from the fluid, ϕ_{wall} decreases in the direction of flow and therefore a_+ and F decrease. From the definition of F, Eq. (211), it follows that the pressure decreases (as was to be expected);

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hence from the Bernoulli equation (210) it follows that u increases. Since c decreases with a decrease in pressure, it follows that the Mach angle ψ also decreases and the + characteristics turn clockwise.

However, if the wall curves in the direction of the fluid, the + characteristics would turn counterclockwise and therefore intersect with each other. This gives rise to a shock wave. If shock waves are to be avoided, the + characteristics can turn clockwise only if the fluid is limited in both directions such that the intersections of the characteristics occur only outside of the fluid. However, in such a case, it should be borne in mind that the solution given above only holds as long as the - characteristics come from the region of uniform flow. The solution breaks down (except in special cases) when the - characteristics start coming from the upper boundary in the disturbed region.

The solution above coincides with the Prandtl-Meier expansion if the wall describes a sharp corner.

The Function F for a Perfect Gas

The function F introduced above can be evaluated analytically if we consider a perfect-gas equation of state. In that case:

$$\int dp/\rho = c^2/(\gamma - 1) \tag{219}$$

and Bernoulli's equation (210) becomes:

$$u^2 = \Pi - 2\,c^2/(\gamma - 1) \tag{220}$$

Hence

$$\frac{1}{\sin^2 \psi} = \frac{u^2}{c^2} = \frac{\Pi}{c^2} - \frac{2}{\gamma - 1} \tag{221}$$

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Furthermore,

$$\frac{dp}{dc} = \frac{dp}{d\rho}\,\frac{d\rho}{dc} = \frac{2\,c\rho}{\gamma - 1} \tag{222}$$

and

$$\frac{dc}{d\psi} = \frac{\cos\psi}{\sin^3\psi}\,\frac{c^3}{\pi} = \frac{c\,\cot\psi}{1 + [2/(\gamma - 1)]\sin^2\psi} \tag{223}$$

Therefore from Eq. (211):

$$\frac{dF}{d\psi} = \frac{dF}{dp}\,\frac{dp}{dc}\,\frac{dc}{d\psi} = \frac{2\cos^2}{\gamma - 1 + 2\sin^2\psi} \tag{224}$$

And this can be integrated to give

$$F = -\psi - \sqrt{\frac{\gamma + 1}{\gamma - 1}}\,\tan^{-1}\left[\sqrt{\frac{\gamma - 1}{\gamma + 1}}\,\cot\psi\right] \tag{225}$$

In this form the function F turns out to be independent of both the constant π and the entropy. However, this is not the case for other equations of state.

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IV. DETONATIONS WAVES. VALIDITY OF CHAPMAN-JOUGUET CONDITION

(Lecture by Peierls)

15. DERIVATION OF DETONATION EQUATIONS

A detonation is a shock wave followed by a chemical reaction which furnishes sufficient energy to maintain stationary conditions at the front. The conservation of mass and the conservation of momentum remain unchanged but of course the energy equation must be modified.

Let + D equal the velocity of the detonation wave. Then, if the solid explosive in front of the detonation wave is initially at rest, $D_1 = -D$. The velocity of the explosive gases behind the detonation front is $U_2 = -D_2 + D$. Here U_2 is positive since the explosive gases move in the same direction as the detonation. The mass of the explosive detonating per unit time per unit cross-sectional area is M. All of the other quantities retain the same significance as in the normal shocks. The equation of conservation of matter remains:

$$M = \frac{D}{V_1} = \frac{D_2}{V_2} \tag{226}$$

The combination of conservation of mass and momentum Equation (56) remains:

$$M^2 = \frac{p_2 - p_1}{V_1 - V_2} \tag{227}$$

And a combination of conservation of mass, momentum, and energy leads again to Eq. (61) or:

$$E_2 - E_1 = \frac{(p_1 + p_2)(V_1 - V_2)}{2} \tag{228}$$

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The only difference then between a shock and a detonation comes in the expressions which we use for $E_2 - E_1$. In the case of a detonation, initially the explosive has an internal energy made up of chemical energy, E_o, and ordinary thermal energy which appears in the equation of state. In going to the final state, the explosive releases its chemical energy. The problem is exceedingly complicated because of the many simultaneous equilibria between C, CO, CO_2, etc., which are established in the explosive gases at the conditions of extremely high densities and pressures, etc. Bright Wilson in the United States and H. Jones in England have made very thorough studies of the equations of state and thermochemistry of the more usual explosives. In order to obtain explicit solutions to the detonation equations it is first necessary to assume a form for the equation of state.

EXAMPLE: Perfect-gas Equation of State.

For the sake of orientation, throwing all attempts at accuracy aside, let us assume that the explosive satisfies the perfect-gas equation of state and E_o is the chemical energy released. Then:

$$E_2 - E_1 = - E_o + \frac{(p_2V_2 - p_1V_1)}{\gamma - 1} \tag{229}$$

For a shock, $E_o = 0$. Combining Eqs. (228) and (229) and remembering that the initial pressure (usually one atmosphere) is negligible with respect to the detonation pressure, p_2, (usually of the order of 200,000 atmospheres):

$$V_2 = \frac{1}{p_2} \frac{2(\gamma - 1) E_o}{\gamma + 1} + \left(\frac{\gamma - 1}{\gamma + 1}\right) V_1 \tag{230}$$

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Or if γ is set equal to 3 as is customary under these conditions:

$$\frac{V_2}{V_1} = \frac{1}{p_2}\,\frac{E_Q}{V_1} + \frac{1}{2} \tag{231}$$

Eq. (231) is an example of the Hugoniot pressure-volume relationship. It expresses V_2 in terms of p_2 for a given initial condition, (p_1, V_1). The hyperbola of Fig. 18 shows this relationship.

It is therefore apparent that the equations of conservation of mass, momentum and energy do not uniquely define the detonation pressure and specific volume. The values of p_2 and V_2 must lie somewhere along the Hugoniot curve. Chapman and Jouguet independently made the hypothesis that if one draws the tangent from the Hugoniot curve through the point (p_1, V_1) the point of tangency on the Hugoniot curve is the point (p_2, V_2). This point is labeled C. J. in Fig. 18. We can show that for this final state, the detonation velocity is just equal to the velocity of sound in the explosive gases relative to the motion of the gases, i.e., $D_2 = C_2$. Von Neumann has shown under what conditions the Chapman-Jouguet hypothesis is valid and under what conditions it fails.

On the Hugoniot diagrams the slope $\tan\varphi$ of a line from p_1, V_1 to any point p_2, V_2 is proportional to the square of the detonation velocity. This can be seen from Eqs. (226) and (227) since $M^2 = \tan\varphi$. Thus:

$$\tan\varphi = \frac{p_2 - p_1}{V_1 - V_2} = \frac{D^2}{V_1^2} \tag{232}$$

The Chapman-Jouguet hypothesis, therefore, leads to the lowest possible detonation velocity. For a smaller angle φ, the line from p_1, V_1 never

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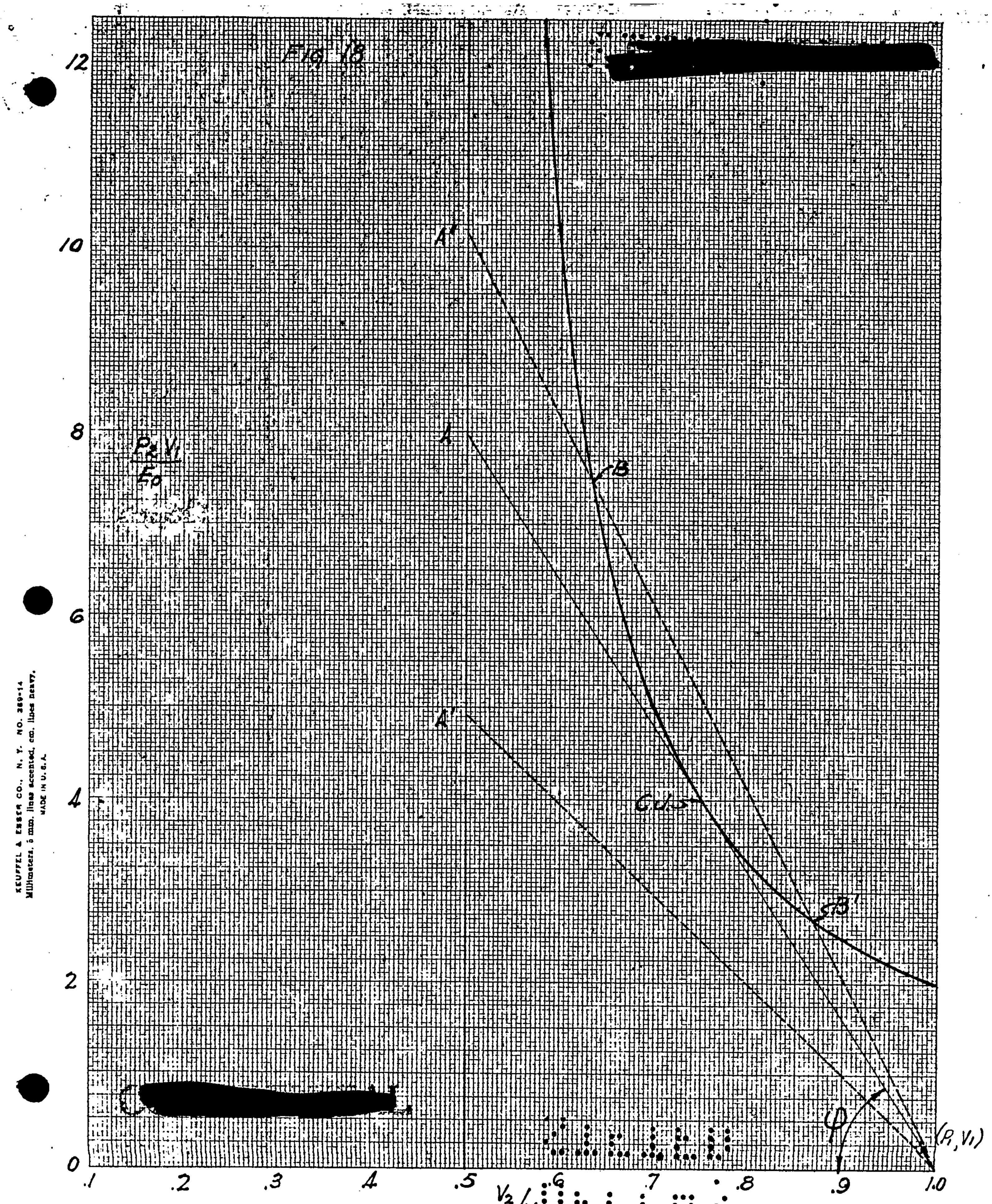

Fig. 18
[redacted]
12
10
8
6
4
2
0
$\frac{P_2 V_1}{E_0}$
A'
A
A'
B
Cusp
B'
φ
(P_1, V_1)
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
V_2/V_1
C [redacted]
KEUFFEL & ESSER CO., N. Y. NO. 359-14
Millimeters, 5 mm. lines accented, cm. lines heavy.
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reaches the Hugoniot curve and this would correspond to the explosion releasing an energy less than E_o.

For the case of the ideal gas with $\gamma = 3$, at the Chapman Jouguet point, $p_2 = 4(E_o/V_1)$ and $V_2 = (3/4)V_1$ so that $D = 4\,(E_o)^{1/2}$, also $C_2 = D_2 = (V_2/V_1)D = (3/4)D = 3(E_o)^{1/2}$. For TNT, $E_o = 1000$ cal/gm $= 41,290$ cc-atm/gm $= 1,400,000$ ft-lb/lb $= 45 \times 10^6/g$ (ft/sec)^2 (the factor g gets absorbed when we use slugs for our mass unit as implied in the above equations). The original density of TNT is 1.70 gm/cc so that $V_1 = .59$ cc/gm. Thus the detonation pressure for TNT should be $p_2 = 4 \times 41,290/.59 = 280,000$ atmospheres and the detonation velocity should be $D = 4 \times (45 \times 10^6)^{1/2} = 27,000$ ft/sec. Both of these values are much too large. Better values could be obtained by taking a smaller value of E_o, but the reason for the discrepancy is that the ideal-gas equation with $\gamma = 3$ is not applicable in the low-pressure region and only approximately true for the very-high-pressure region.

Another property of the Chapman-Jouguet point is that it has the maximum entropy of any point along the Hugoniot curve. Consider $E_2(V_2, S)$ and remember from Eqs. (12) and (13) and thermodynamics that $(\partial E_2/\partial V_2)_S = -p_2$ and $(\partial E_2/\partial S)_{V_2} = T$, so that:

$$\frac{dE_2}{dV_2} = \left(\frac{\partial E_2}{\partial V_2}\right)_S + \left(\frac{\partial E_2}{\partial S}\right)_{V_2} \frac{dS}{dV_2} = -p_2 + T\,\frac{dS}{dV_2} \qquad (233)$$

Therefore if we take the derivative of both sides of Eq. (228) with respect to V_2 keeping p_1, V_1, and E_1 constant, we get:

$$\frac{dE_2}{dV_2} = -\,p_2 + T\,\frac{dS}{dV_2} = -\,\frac{p_1 + p_2}{2} + \frac{1}{2}\,(V_1 - V_2)\frac{dp_2}{dV_2} \qquad (234)$$

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or

$$T\frac{dS}{dV_2} = \frac{V_1 - V_2}{2}\left[\frac{p_2 - p_1}{V_1 - V_2} + \frac{dp_2}{dV_2}\right] \tag{235}$$

Therefore, dS/dV_2 is zero and the entropy has a stationary value (along the Hugoniot curve) if either $V_1 = V_2$ or else we are at the Chapman-Jouguet point where the slope of the Hugoniot is

$$-\frac{dp_2}{dV_2} = \frac{p_2 - p_1}{V_1 - V_2} \tag{236}$$

The condition that $V_1 = V_2$ corresponds to a minimum entropy and the Chapman-Jouguet condition corresponds to a maximum entropy. Until quite recently the only arguments advanced for the impossibility of V_2 being greater than the Chapman-Jouguet specific volume was based on the smaller entropy of such points leading to instability. However, such arguments were not convincing and it remained for von Neumann to prove the impossibility of such points on the basis of kinematical arguments.

Since dS/dV_2 is zero in the vicinity of the Chapman-Jouguet point, it follows that for this final state the velocity of sound is given by the relation:

$$c_2^2 = \left(\frac{\partial p_2}{\partial \rho_2}\right)_S = -V_2^2\left(\frac{dp_2}{dV_2}\right) = +V_2^2\left(\frac{p_2 - p_1}{V_1 - V_2}\right) \tag{237}$$

and by virtue of Eqs. (232) and (226):

$$c_2^2 = \frac{V_2^2}{V_1^2} D^2 = D_2^2 \tag{238}$$

[redacted]

[redacted]

or

$$C_2 = D_2 \tag{239}$$

Thus, if the final state corresponds to the Chapman-Jouguet condition, the velocity of the explosive gases relative to the detonation wave is just equal to the velocity of sound.

The arguments of von Neumann depend on the fact that if we have an equation of state for which the Hugoniots look like Figure 19 (a) only compressional shocks are stable. Whereas if we have a pathological Hugoniot such as shown in Figure 19(b) only rarefaction shocks are stable. This may be seen by dividing the supposed shock into two parts. If the first part travels faster than the second part, the shock is unstable and will divide itself up into many small changes. However, if the second part of the shock travels faster than the first part, the shock will maintain itself and show no tendency to split up into smaller shocks.

The velocity, U, of a shock wave which goes between any point p_1, V_1 to a point p_2,V_2 on the same Hugoniot curve is given by a relation similar to that of Eq. (232). If the medium in front of the shock is at rest, then the same arguments which led to Eq. (232) apply (with $E_o = 0$) and

$$U^2 = V_1^2 \frac{p_2 - p_1}{V_1 - V_2} = V_1^2 \tan \varphi \tag{240}$$

Therefore the greater the slope φ the greater the shock-wave velocity.

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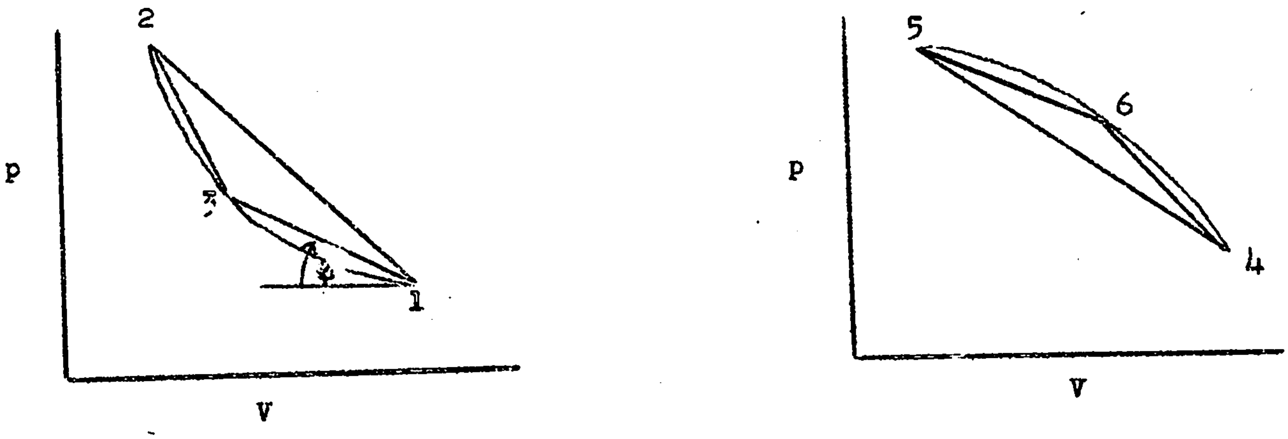

19(a) 19(b)

Figure 19

1) Compressional shock in Figure 19(a). Consider a compressional shock going from point 1 to point 2. If we tried to break this up into two smaller shocks one going from 1 to 3 followed by one going from 3 to 2, we would notice that ϕ is less for the shock from 1 to 3 than for the shock from 3 to 2. This means that the 3-to-2 shock will travel faster and overtake the 1-to-3 shock. So this compressional shock is stable.

2) Rarefaction shock in Figure 19(a). The argument for the instability of the rarefaction shock for a Hugoniot such as shown in Figure 19(a) proceeds as before. However, now the 2-to-3 shock is spatially in front of the 3-to-1 shock and therefore the faster velocity of the 2-to-3 tends to separate the two shocks. Thus a rarefaction shock will tend to decompose.

3) Compressional shock in Figure 19(b). If we divide the compressional shock 4-to-5 up into two smaller shocks 4-to-6 followed by 6-to-5, we notice that the slope ϕ is larger for the 4-to-6 than for the 6-to-5. Thus the

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4-to-6 will travel faster and run away from the 6-to-5 shock. This means that for this pathological Hugoniot, the compressional shock would be unstable.

4) Rarefaction shock in Figure 19(b). Since a shock from 5-to-6 would travel slower than a shock from 6-to-4, it follows that small rarefaction shocks will tend to combine to produce larger discontinuities. Thus rarefaction shocks are stable for the pathological Hugoniot shown in Figure 19(b).

Von Neumann postulates that a detonation is made up of two separate steps.[17] In the very front of the detonation wave, the material is highly compressed by a shock but no chemical reaction has taken place. Directly behind the detonation front comes the reaction zone in which the chemical reactions take place. Experimentally it is known that the reaction zone extends over a distance of between a fraction and a few centimeters depending on the explosive. In order for the shape of the detonation wave to be independent of time, it is necessary that the initial compression shock and the subsequent rarefactions (during which the chemical reactions proceed) must both be stable and travel at the same velocity. This is a stringent condition and serves to limit the possible final states of the explosive gases.

If E_o is the final amount of chemical energy liberated in the explosion and $n E_o$ is the amount of energy already liberated at any time, then we can follow the course of the detonation by drawing a sequence of Hugoniots for different values of n varying from $n = 0$ before the reactions have started to $n = 1$ where the reactions have been completed.

17) J. von Neumann, OSRD No. 549

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The normal shape of these curves is shown in Figure 20.

The first step in the detonation is the compression shock which takes the explosive from p_1, V_1 to some point A, A', or A" on the n = 0 Hugoniot. The slope, $\tan \varphi = (p_2 - p_1)/(V_2 - V_1)$, where 2 may represent A, A', or A" determines the shock velocity. By the conservation of mass, momentum, and energy it follows that any changes of pressure and volume which take place at this velocity must have this same slope. Thus, for a stationary detonation wave, it follows that if the initial compression shock has taken the explosive to some point A, A', or A" on the n = 0 Hugoniot, the states reached during the subsequent chemical reactions and rarefactions must lie along the line joining p_1, V_1 with this point.

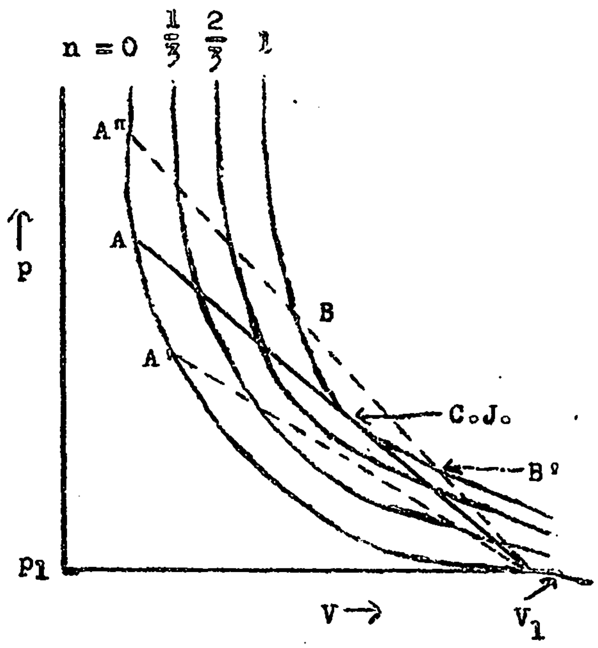

Figure 20

Thus it is apparent that the initial compression could not carry us to the point A' because in that case, the line p_1, V_1 to A' does not intersect the n = 1 Hugoniot and therefore the chemical reactions could not go to completion without forcing the detonation velocity to be larger.

The line p_1, V_1 to A which passes through the Chapman-Jouguet point has the smallest slope which is possible from this standpoint.

Suppose that the initial compression shock leads to the point A". Then as the chemical reactions proceed, we can make gradual

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rarefactions until we reach the point B. To reach the point B' from B would involve an unstable rarefaction shock which would soon break up into gradual rarefactions along the Hugoniot from B to the Chapman-Jouguet point. Since the entropy at the Chapman-Jouguet point is a maximum along the Hugoniot, it would be impossible for the rarefaction to proceed further. Thus it is possible kinematically to reach any point at or above the Chapman-Jouguet point. And it is impossible to reach any point below the Chapman-Jouguet point. Whether the rarefactions will proceed from a point B to the Chapman-Jouguet point depends on the conditions behind the detonation front. The velocity with respect to the detonation front of the explosive gases at the Chapman-Jouguet point is just equal to the velocity of sound. But for points, B, above the Chapman-Jouguet point it may be shown that the velocity of the explosive gases is subsonic. If the pressure continues to decrease behind the detonation wave, the velocity of the explosive gases right behind the detonation wave is supersonic with respect to the gases further back. In this case, the gases further back cannot send disturbances up to the detonation front to oppose the rarefaction from B to the Chapman-Jouguet point. That this change will occur spontaneously if not opposed is guaranteed by the increase in entropy. However, if the pressure should rise behind the detonation front, the velocity of the explosive gases right behind the detonation wave remains subsonic with respect to the gases farther behind and disturbances can travel up to the detonation front and maintain a point above the Chapman-Jouguet point. Thus, in the normal case, we expect the Chapman-Jouguet point. But we can set up special examples such as a detonation from a high explosive setting

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off a detonation in a weak explosive where the Chapman-Jouguet condition would not apply. Another example is a spherically converging detonation wave (see J. M. Keller, LA-143).

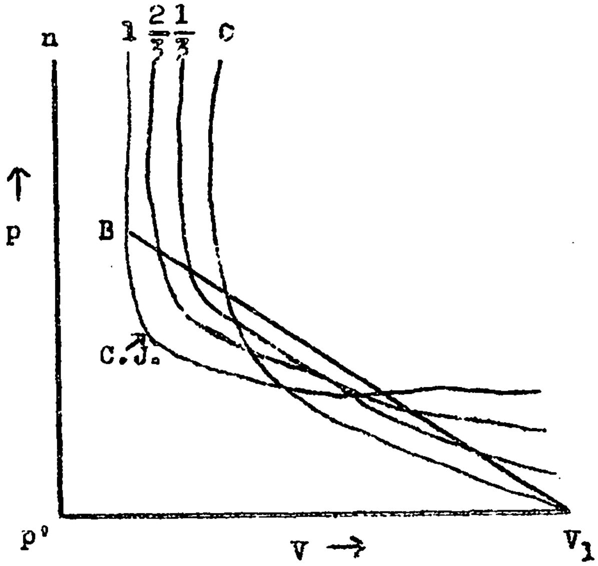

Figure 21

We can imagine another situation where the Chapman-Jouguet condition would not apply. In Figure 20 we have supposed that the Hugoniot curves for constant n do not cross. If they should cross it would be impossible to proceed gradually from a point A" to a point B. Since the course of the chemical reactions must proceed along an orderly path from $n = 0$ to $n = 1$, it would be necessary to pass through unstable rarefaction shocks. The only possible final states that may be reached under those conditions correspond to no crossings of the various Hugoniots between A" and B. Unless this is possible, no stationary detonation wave is possible for the system. If $(\partial p_2/\partial n)_{V_2}$ is greater than zero for all values of n and V, this difficulty cannot arise. This implies that $(\partial E_2/\partial n)_{p_2, V_2}$ is positive or that each stage of

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the explosive reaction is exothermic[14]. In Fig. 21, a system of crossing Hugonoits is shown. Here the previous arguments indicate that the Chapman-Jouguet point is not possible, but any point at B or above might represent a stable detonation wave.

16. PLANE DETONATION WAVE INITIATED FROM FIXED WALL ✓

Consider a detonation wave proceeding in the x direction from a fixed wall at $x = 0$. We shall assume that the Chapman-Jouguet condition holds and the explosive satisfies the perfect gas equation with $\gamma = 3$. Under these conditions, at the detonation front:

$$C_2 = D_2 = \frac{3}{4} D \tag{241}$$

$$U_2 = D - D_2 = \frac{1}{4} D \tag{242}$$

The velocity of the explosive gases relative to the wall varies between zero at the wall and $C_2/3$ at the detonation front. Therefore we can

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14) Consider E_2 as a function of n, p_2, and V_2. Then differentiate both sides of Eq. (228) with respect to n keeping p_2, p_1, and V_1 constant

$$\left(\frac{\partial E_2}{\partial n}\right)_{p_2,V_2} + \left(\frac{\partial E_2}{\partial V_2}\right)_{n_2,p_2}\left(\frac{\partial V_2}{\partial n}\right)_{p_2} = -\frac{1}{2}(p_1 + p_2)\left(\frac{\partial V_2}{\partial n}\right)_{p_2}$$

But $(\partial E_2)/(\partial V_2)_{n_2,p_2} = -p_2$ so that rearranging the above equation:

$$\frac{1}{2}\left(\frac{\partial V_2}{\partial n}\right)_{p_2}(p_2 - p_1) = \left(\frac{\partial E_2}{\partial n}\right)_{p_2,V_2}$$

Since $p_2 - p_1$ is always positive, $(\partial V_2)/(\partial n)_{p_2}$ and $(\partial E)/(\partial n)_{p_2,V_2}$ must have the same sign.

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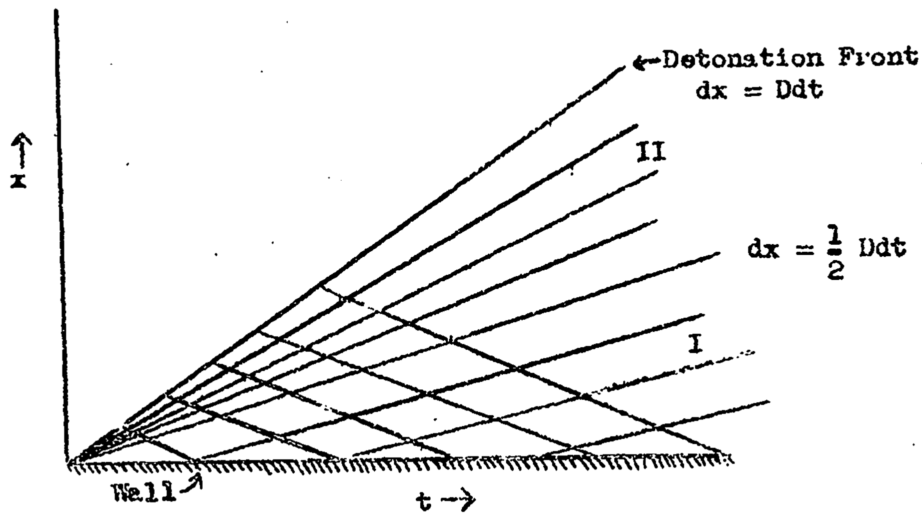

Figure 22

apply the Riemann method to this problem. Since $\gamma = 3$, $\sigma = 0$. The lines of slope $u - c$ conserve the property $u - c$ and the lines of slope $u + c$ conserve the property $u + c$. Along the wall, we require that $u = 0$. Therefore, all of the lines of slope $u - c$ proceeding from the wall have the characteristic value $-c$. But along the detonation front, $u - c = -D/2$. Therefore $(c)_{wall} = D/2$. However, the lines of constant $u + c$ originating on the wall do not extend all of the way to the detonation wave since these $u + c$ have the slope $D/2$. Thus in region I, lying between the wall and $dx = \frac{1}{2} D\,dt$ the velocity, u, is zero and the velocity of sound is $\frac{1}{2}$ D.

The other lines of constant $u + c$ cannot originate on the detonation front, since it itself is a line of constant $u + c$. Thus the remaining lines of constant $u + c$ must originate at the origin. They must be straight lines since the lines of constant $u - c$ which they cross all have have the same value. The slope, dx/dt, of these lines of constant $u + c$ is also $u + c$. Therefore in region II:

$$\frac{dx}{dt} = \frac{x}{t} = u + c \qquad (243)$$

But

$$u - c = -\frac{D}{2} \qquad (244)$$

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Therefore:

Region II

$\frac{D}{2} < \frac{x}{t} < D$

$$\begin{cases} u = \frac{x}{2t} - \frac{D}{4} & (245) \\ c = \frac{x}{2t} + \frac{D}{4} & (246) \end{cases}$$

Region I

$0 < \frac{x}{t} < \frac{D}{2}$

$$\begin{cases} u = 0 & (247) \\ c = \frac{D}{2} & (248) \end{cases}$$

This forms a complete solution to the problem.

If the Chapman-Jouguet condition were not satisfied and the final state corresponded to a higher pressure and a lower specific volume than the Chapman-Jouguet point, then $u + c$ is less than D and lines of constant $u + c$ cross the detonation front. Under such conditions, the conditions at the front are affected by the conditions in the rear and the detonation velocity must be adjusted to fit these conditions.

17. PLANE DETONATION INITIATED WITH FREE SURFACE[15]

Consider a detonation wave proceeding from a free surface at $x = 0$. We shall assume again that the Chapman-Jouguet condition is satisfied and the explosive satisfies the perfect-gas equation with $\gamma = 3$. This problem is very similar to the case with the fixed wall. Again, the lines of constant $u - c$ cut across the detonation front. Since $u = D/4$ and $c = 3D/4$ right behind the detonation front, it follows that all of the lines of constant $u - c$ have the characteristic value $-D/2$. All of the

15) G. I. Taylor, BM-49, AC-639

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lines of constant $u + c$ originate at the origin and are straight lines with the slope

$$x/t = u + c \tag{249}$$

Therefore just as in region II of the previous problem:

$$u = \frac{x}{2t} - \frac{D}{4} \tag{250}$$

$$c = \frac{x}{2t} + \frac{D}{4} \tag{251}$$

The free surface has a pressure and hence density and velocity of sound equal to zero. It would therefore have the equation:

$$\text{(free surface)} \qquad x/t = - D/2 \tag{252}$$

Along the free surface, $u + c = D/2$ so that the free surface is itself a line of constant $u + c$. Figure 23 illustrates the problem.

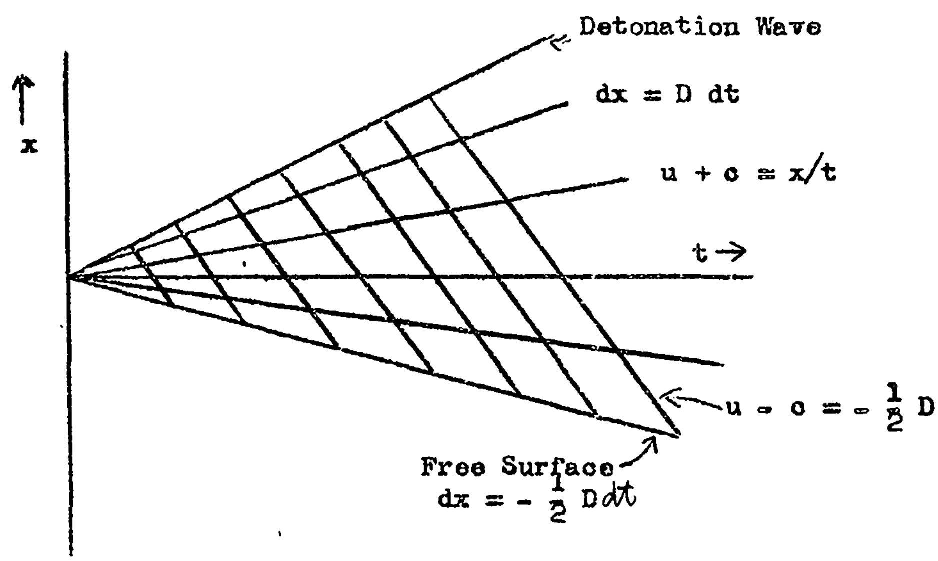

Figure 23

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18. SPHERICALLY DIVERGING DETONATION WAVES[16]

Spherically diverging detonation waves are examples where the Chapman-Jouguet condition applies. The following treatment is due to G. I. Taylor.

If we suppose that the velocity of the expanding detonation wave is radial and has the magnitude u, then the equation of motion (96) becomes in spherical coordinates:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = \frac{1}{\rho} \frac{\partial p}{\partial r} \tag{253}$$

And the equation of continuity (97) becomes[17]:

$$\frac{\partial \rho}{\partial t} = - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u) = -2 \frac{\rho u}{r} - \frac{\partial}{\partial r} (\rho u) \tag{254}$$

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16) G. I. Taylor, BM-49, AC-639

17) If $\underline{r_1}$ is the unit vector in the radial direction, it may be expressed in rectangular coordinates:

$$\underline{r_1} = \underline{i}\,(x/r) + \underline{j}\,(y/r) + \underline{k}\,(z/r)$$

Thus the equation of continuity is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u \underline{r_1}) = - \frac{\partial}{\partial x}\left(\rho u \frac{x}{r}\right) - \frac{\partial}{\partial y}\left(\rho u \frac{y}{r}\right) - \frac{\partial}{\partial z}\left(\rho u \frac{z}{r}\right)$$

$$= -3 \frac{\rho u}{r} - \left[x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z}\right] \frac{\partial}{\partial r}\left(\frac{\rho u}{r}\right) = - \frac{3\rho u}{r} - r \frac{\partial}{\partial r}\left(\frac{\rho u}{r}\right)$$

$$= -2 \frac{\rho u}{r} - \frac{\partial}{\partial r} (\rho u) = - \frac{1}{r^2} \frac{\partial}{\partial r} (\rho u r^2)$$

[redacted]

The equation of motion is then the same as for the one-dimensional plane case, but the equation of continuity has the additional term $- 2\rho u/r$. We seek solutions of the equations such that u, p, ρ, and c are all functions only of $a = r/t$. This would mean that:[18]

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial r} = 0$$

$$\frac{\partial p}{\partial t} + a\frac{\partial p}{\partial r} = 0 \qquad (255)$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial r} = 0$$

So that the equation of motion (253) becomes

$$(u - a)\ du/da = -(1/\rho)\ dp/da \qquad (256)$$

And the equation of continuity becomes:

$$\frac{(u - a)}{\rho}\frac{d\rho}{da} + \frac{du}{da} + \frac{2u}{a} = 0 \qquad (257)$$

But since $c^2 = dp/d\rho$, $dp/da = c^2\ d\rho/da$ so that Eqs. (256) and (257) may be combined to give:

$$-\frac{u - a}{\rho}\frac{d\rho}{da} = \frac{du}{da} + \frac{2u}{a} = \left(\frac{u - a}{c}\right)^2\frac{du}{da} \qquad (258)$$

Or eliminating $d\rho/da$:

$$\left[1 - (u - a)^2/c^2\right] du/da = - 2u/a \qquad (259)$$

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18) This follows since:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial r} = \left[\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial r}\right]\frac{du}{da} = \left[-\frac{a}{t} + \frac{a}{t}\right]\frac{du}{da} = 0$$

In Eq. (258) it is convenient to represent $(1/\rho)d\rho/da$ as $(1/\rho)(d\rho/dc^2)dc^2/da$ since c is a function only of the density and the density may be regarded as a function only of c. Thus letting

$$f = \frac{\rho}{c^2}\,\frac{dc^2}{d\rho} \tag{260}$$

it follows that:

$$\frac{c^2 f}{\rho}\,\frac{d\rho}{da} = \frac{dc^2}{da} \tag{261}$$

For a perfect gas, $f = \gamma - 1$ (262)

These equations may be set in dimensionless form by replacing the variables u and c by the dimensionless quantities:

$$\alpha = u/a \tag{263}$$

$$\beta = u/c \tag{264}$$

$$Z = \ln a/a_o \tag{265}$$

Here a_o is the value of a for the radius of detonation.

Combining Eqs. (255), (259), and (260) we get:

$$\frac{d\alpha}{da} = -\frac{\alpha}{a}\,\frac{3\alpha^2 - \beta^2(\alpha - 1)^2}{\alpha^2 - \beta^2(\alpha - 1)^2} \tag{266}$$

$$\frac{d\beta}{da} = -\frac{2\alpha}{a\beta^2}\,\frac{\alpha^2 - f\beta^2\alpha(\alpha - 1) - \beta^2(\alpha - 1)^2}{\alpha^2 - \beta^2(\alpha - 1)^2} \tag{267}$$

$$\frac{dZ}{da} = \frac{1}{a} \tag{268}$$

Combining Eqs. (266) and (268), it follows that:

$$\frac{d\alpha}{dZ} = -\alpha\,\frac{3\alpha^2 - \beta^2(\alpha - 1)^2}{\alpha^2 - \beta^2(\alpha - 1)^2} \tag{269}$$

And combining Eqs. (267) and (268):

$$\frac{d\beta}{dZ} = -\frac{2\alpha}{\beta^2}\,\frac{\alpha^2 - f\beta^2\alpha(\alpha - 1) - \beta^2(\alpha - 1)^2}{\alpha^2 - \beta^2(\alpha - 1)^2} \qquad (270)$$

To solve these equations for any particular case it is first necessary to know the equation of state. We then proceed as follows:

1). First calculate c^2 and $dc^2/d\rho$ as functions of ρ. Next tabulate f as a function of c.

2). The Chapman-Jouguet condition and the Hugoniot relations are the same as for the case of the plane detonation. They determine the detonation velocity, D, and the value of $u = D - D_2 = D(1 - 1/\eta)$ at the shock wave front. (Notice that u corresponds to the u_2 of the plane detonation case.) If R is the radius of the detonation wave at any time, then from the form of the solution which we require:

$$R = D\,t \qquad (271)$$

and at any other point at this time, $r = a\,t$. Therefore we have the similarity condition:

$$a/D = r/R \qquad (272)$$

Therefore it follows that:

$$u/D = \alpha\, r/R \qquad (273)$$

$$c/D = \alpha r/\beta R \qquad (274)$$

$$Z = \ln(r/R) \qquad (275)$$

But right behind the detonation front, where $r = R$, according to the Eq. (273) and the conservation of matter (226):

$$\alpha = u/D = 1 - 1/\eta \qquad (276)$$

And the Chapman-Jouguet condition gives: $u + c = D$ so that from Eqs. (273) and (274) for $r = R$:

$$1 = \frac{u}{D} + \frac{c}{D} = \alpha + \frac{\alpha}{\beta} = \alpha\left(1 + \frac{1}{\beta}\right) \tag{277}$$

or

$$\beta = \frac{1}{(1/\alpha) - 1} = \frac{\alpha}{1 - \alpha} = [illegible] - 1 \tag{278}$$

The relations (275), (279), and (278) give the values of the variables α, β, and Z at the detonation wave. Then Eqs. (269) and (270) can be integrated numerically to give the conditions at any other point behind the detonation wave.

Just as in the case of the plane detonation wave initiated at a fixed wall, the velocity goes to zero at a point between the detonation front and the center (In the case of TNT with initial density of 1.51, H. Jones has computed that this point occurs at $r = .418\ R$. This is to be compared with the value $x = .577\ x_{det}$ which he calculated for the plane wave problem.). The pressure and velocity behind the spherically diverging detonation wave decrease much more rapidly than for the case of the plane waves. Figures 24 and 25 show H. Jones' results for the TNT.

19 SPHERICALLY CONVERGING DETONATION WAVES

Keller (LA-143) has considered the case of spherically converging detonation waves. This problem is much more difficult than the case of the spherically diverging detonation waves. No stationary solution is possible. At first, the Chapman-Jouguet condition gives the detonation velocity and the pressure and velocity right behind the detonation front.

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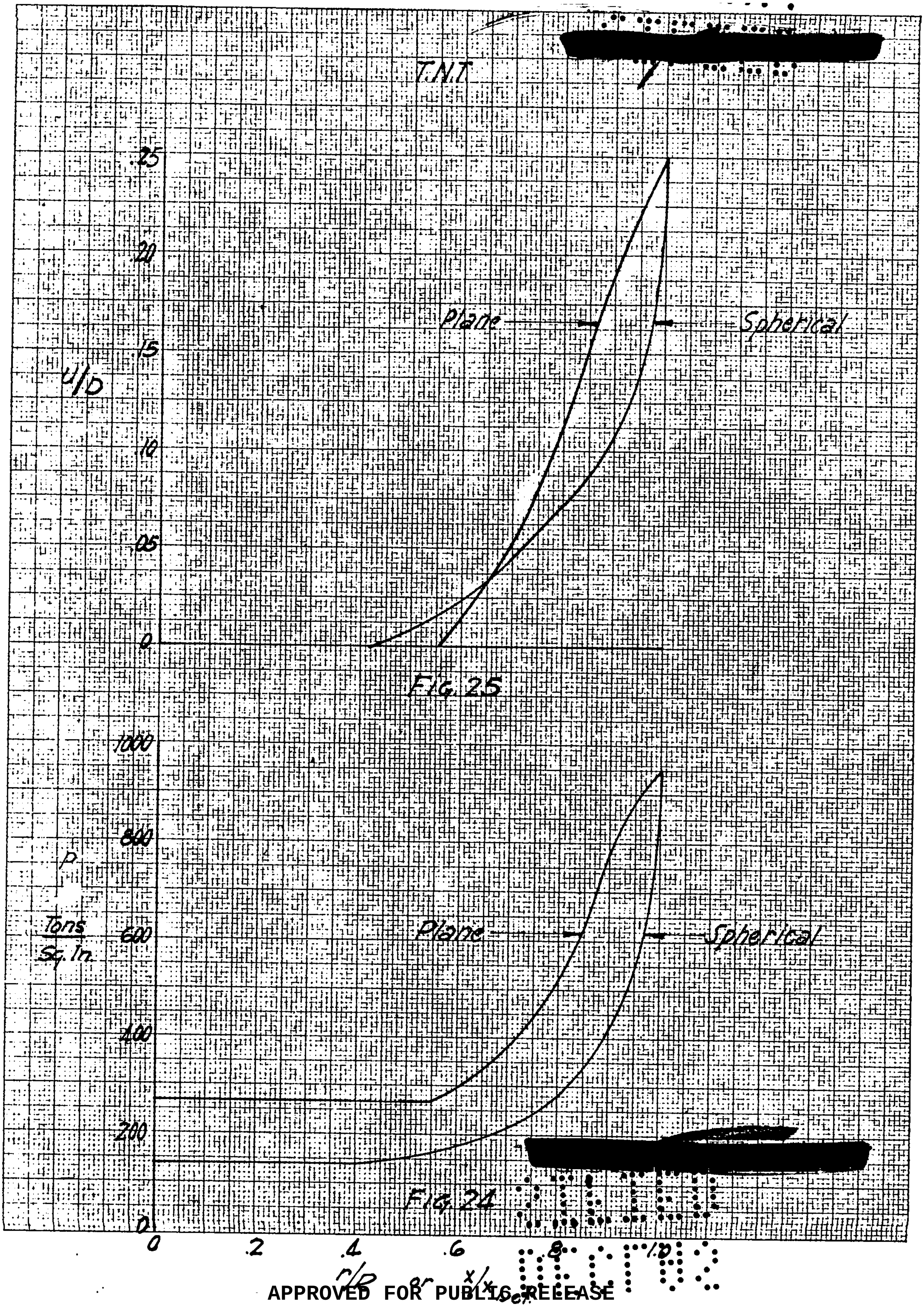

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Gradually at first, rapidly later, the pressure and velocity at the detonation front rises and the detonation velocity increases. By the time the detonation waves have travelled half way to the center, these effects become very important. No analytical solution has been obtained and it was necessary (even in the case of a perfect gas with $\gamma = 3$) to make the calculations with the help of the I.B.M. machines.

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V. SHOCK WAVES

(Applications to Spherical Blast Waves, etc.)

Lectures by Penney

(20). GENERAL EQUATIONS AND VIEWPOINT.

Blast waves are good examples of the shock waves which we studied in a previous section. We are usually interested in blast waves in either air or water. In either case, the perfect-gas equations suffice. For air, $\gamma = 1.4$ and for water, $\gamma = 3$. Usually the matter in front of the shock wave is at rest so that $u_1 = 0$.

Let us summarize the shock-wave equations which are applicable for the blast waves:

$$u_1 = D_1 + U = 0 \qquad \text{or} \qquad U = - D_1 \tag{279}$$

$$u_2 = D_2 + U \tag{280}$$

From Eqs. (56) and (52)

$$M^2 = p_1\rho_1 (\xi-1)\eta/(\eta-1) = D_1^2 \rho_1^2 = D_2^2 \rho_2^2 \tag{281}$$

Here, of course, $\eta = \rho_2/\rho_1$ and $\xi = p_2/p_1$

From Eq. (130):

$$c^2 = \gamma p/\rho \tag{282}$$

and from Eq. (67)

$$\eta = \frac{(\gamma-1) + (\gamma+1)\xi}{(\gamma+1) + (\gamma-1)\xi} \tag{283}$$

Combining Eqs. (279), (281), (282), and (283)

$$\frac{U^2}{c_1^2} = \left(\frac{\rho_1}{\gamma p_1}\right) \frac{p_1}{\rho_1} \frac{\eta(\xi-1)}{(\eta-1)} = \frac{1}{2\gamma} \left[(\gamma-1) + (\gamma+1)\xi\right] \tag{284}$$

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From Eqs. (280), (281), and (283):

$$\frac{U_2}{U} = \frac{D_2+U}{U} = 1 - \frac{D_2}{D_1} = 1 - \frac{\rho_1}{\rho_2} = 1 - \frac{1}{\eta} = \frac{2(\xi-1)}{(\gamma-1)+(\gamma+1)\xi} \qquad (285)$$

Furthermore, it is convenient to discuss the temperature, T_2', to which the gas returns after the wave has passed and the pressure returns to its initial value, p_1. From the perfect-gas law, $pV = RT$, and the perfect-gas adiabat, $pV^\gamma = k(S)$, it follows that:

$$T_2'/T_1 = V_2'/V_1 = (V_2/V_1)\ (p_2/p_1)^{1/\gamma} = \xi^{1/\gamma}/\eta \qquad (286)$$

From Eq. (72) it follows that the ratio of T_2' to T_1 is connected with the change in entropy by the relation:

$$\Delta S = C_p \ln T_2'/T_1 \qquad (287)$$

Instead of considering T_2' itself, we can consider the increase of temperature, θ, such that:

$$T_2' = T_1 + \theta \qquad (288)$$

For weak shocks, it follows from Eq. (73) that:

$$\theta/T_1 = (1/2\gamma)\ (1-1/\gamma^2)\ (\xi-1)^3 + \ldots \qquad (289)$$

Since the change in entropy is equal to the heat dissipated divided by the temperature, it follows that for weak shocks where

$$\Delta S = C_p \ln\left[1 + \theta/T_1\right] = C_p\ \theta/T_1 + \ldots \qquad (290)$$

the heat dissipated in the specific volume, $V_1 = 1/\rho_1$, is $T_1 \Delta S = C_p \theta$. Therefore the heat, H, dissipated by the blast wave in passing through a unit volume of matter is

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$$H = \rho_1 T_1 \Delta S = \rho_1 C_p \theta \qquad (291)$$

For air with $\gamma = 1.4$ and $T_1 = 273^\circ K$, we get:

ξ	2	3	5	10	50	100	200
$\theta(^\circ K)$	2.5	10.3	31	97	579	1065	1417

The energy dissipated per unit volume of air is therefore small if ξ is less than 10, but very large for higher compressions.

For water at 273°K, if we express p_2 in kilobars (approximately 1000 atmospheres) it has been found that:

$$\theta = .0335\, p_2^{3} - .0118\, p_2^{4} + .0035\, p_2^{5} - .01\,0\, p_2^{6} \qquad (292)$$

so that even at a distance of 2 charge radii where $p_2 = 20$ kilobar, $\theta = 20^\circ K$. Or at a distance of 8 charge radii where $p_2 = 1$ kilobar, $\theta = .0252^\circ K$. Thus we can neglect the change of the resultant temperature in all underwater-blast problems.

For very strong blast waves, the hydrodynamic equations become very simple. For example for air with $\gamma = 1.4$, Eqs. (283), (284), and (285) become:

$$\eta = 6 \;;\; U^2/c_1^2 = 6\xi/7 \;;\; u_2/U = 5/6 \qquad (293)$$

For weak shock waves with ξ less than 2, the shock equations can be expanded in powers of $(\xi - 1)$. According to Eq. (68):

$$\eta = 1 + (1/\gamma)(\xi - 1) - (1/2\gamma)(1 - 1/\gamma)(\xi - 1)^2 + \ldots \qquad (294)$$

Then (from Eq. 282),

$$\frac{c_2^2}{c_1^2} = \frac{p_2}{p_1}\,\frac{\rho_1}{\rho_2} = \frac{\xi}{\eta} = 1 + (1 - 1/\gamma)(\xi - 1) - (1/2\gamma)(1 - 1/\gamma)(\xi - 1)^2 + \ldots \qquad (295)$$

[redacted]

[redacted]

from Eq. (284),

$$\frac{U^2}{c_1^2} = 1 + \frac{\gamma+1}{2\gamma}(\xi - 1) \tag{296}$$

and from Eq. (285),

$$\frac{u_2}{U} = \frac{1}{\gamma}(\xi - 1) - \frac{\gamma + 1}{2\gamma^2}(\xi - 1)^2 + \dots \tag{297}$$

(21). DEGRADATION OF STRONG BLAST WAVES (NOT NEGLECTING ENERGY DISSIPATION).

For strong blast waves in air, it is not possible to neglect the dissipation of energy and the resulting entropy gradient in the air behind the blast wave. The hydrodynamical equations in spherical coordinates then become somewhat too complicated to solve analytically. Penney[19)] has developed an extension of the Riemann method which is applicable to this case. He introduces two functions P and Q defined by the relations:

$$P = \sigma + u \tag{298}$$

$$Q = \sigma - u \tag{299}$$

Here u is the radial velocity of the gas (assuming that its motion is strictly radial) and σ is the usual Riemannian variable:

$$\sigma = \int_0^\rho \frac{c}{\rho}\, d\rho \tag{300}$$

where

$$c^2 = (\partial p/\partial \rho)_\Theta \tag{301}$$

since a constancy of Θ implies the constancy of entropy.

Making use of Eqs. (253) and (254) for the equation of motion and equation of continuity in spherical coordinates and making use of the fact that Θ remains constant for a given gas particles, i.e., $\partial\Theta/\partial t + u\ \partial\Theta/\partial r = 0$, it follows after a considerable job of algebraic manipulations that:

19) W. G. Penney, (BM-37; RC-260)

[redacted]

[redacted]

1) For a point moving with the speed $dr/dt = u + c$,

$$dP = dt\left[-2uc/r + c(\partial\sigma/\partial\theta)_p\ (\partial\theta/\partial r)_t\right] \tag{302}$$

2) For a point moving with the speed $dr/dt = u - c$,

$$dQ = dt\left[-2uc/r - c(\partial\sigma/\partial\theta)_p\ (\partial\theta/\partial r)_t\right] \tag{303}$$

Thus P and Q serve the same purpose in the blast wave calculations as the lines of constant $\sigma + u$ and constant $\sigma - u$ in the one-dimensional Riemann method. The only difference is that now the line with a slope of $u + c$ will have a value of P which varies slowly with time. These quantities lend themselves to a point-by-point numerical integration such as indicated in Lecture I, Section 2. For a perfect gas, $\sigma = 2c/(\gamma - 1)$, and the expressions for dP and dQ, Eqs. (302) and (303), become:

1) For a point moving with the speed $dr/dt = u + c$

$$dP = dt\left[-\frac{2uc}{r} + \frac{c^2}{\gamma-1}\ \frac{\partial}{\partial r}\ \left(\ln\left(\frac{T_1+\theta}{T_1}\right)\right)\right] \tag{302'}$$

2) For a point moving with the speed $dr/dt = u - c$,

$$dQ = dt\left[-\frac{2uc}{r} - \frac{c^2}{\gamma-1}\ \frac{\partial}{\partial r}\ \left(\ln\left(\frac{T_1+\theta}{T_1}\right)\right)\right] \tag{303'}$$

The boundary conditions are those for shock waves summarized in Eqs. (279) through (285) or Eq. (293).

The only tricky feature of the calculations is that the lines with the characteristic, P, are generated with the speed $dr/dt = u + c$ which usually is faster than the velocity of the blast wave. The characteristics in front of the shock wave must be disregarded.

In this way, Penney has found (in agreement with experiment) that the peak overpressure, $p_2 - p_1 = p_{max}$, in lbs/in^2, produced by the explosion

[redacted]

[redacted]

of W lbs of high explosive at a distance R feet, may be represented by the relation:

$$p_{max} = \frac{AW^{1/3}}{R} \, e^{BW^{1/6}/R^{1/2}} \tag{304}$$

Experimental values of the constants A and B for various explosives are:

Explosive	A	B
TNT (cast)	10.7	7.1
Torpex	12.3	6.3
Dithekite	12.0	5.6

This peak pressure decreases much more rapidly than would be expected from acoustical theory. In a distance of between 8 and 20 charge radii the peak pressure falls a factor of 9 as compared to a factor of 2 on the basis of the theory of sound. The peak pressure at 20 charge radii is around 4 atmospheres.

(22). SHAPE OF BLAST WAVE AT LARGE DISTANCES.

At large distances where the peak pressure is less than twice the initial pressure (ξ less than 2), we can neglect the energy degradation and θ. In that case it is easy to show in a rough qualitative fashion that the peak pressure should decrease inversely proportionally to the distance, R. Using the method of the last section, we have:

$$\sigma = \frac{2}{\gamma - 1} (c - c_1) \tag{305}$$

The additive constant, $2c_1/(\gamma - 1)$, is added for convenience. It is easy to show that any constant number added to σ cannot affect the conditions in the fluid. Then right behind the blast wave (using Eqs. (299), (295), (296) and (297)),

[redacted]

$$Q = \sigma - u_2 = \frac{2}{\gamma-1}\,(c_2 - c_1) - \frac{u_2}{U}\,\frac{U}{c_1}\,c_1$$

$$= \frac{2c_1}{\gamma-1}\left[\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)(\xi-1) - \left(\frac{\gamma^2-1}{8\gamma^2}\right)(\xi-1)^2 + \cdots\right] \tag{306}$$

$$- c_1\left[\frac{1}{\gamma}(\xi-1) - \frac{\gamma+1}{2\gamma^2}(\xi-1)^2 + \cdots\right]\left[1 + \frac{\gamma+1}{4\gamma}(\xi-1) + \cdots\right]$$

$$= 0 \text{ through terms of the order of } (\xi-1)^2$$

Also it may be shown that at points behind the blast front, Q remains practically zero. This follows from the fact that

$$dQ/dt = -2uc/r \qquad \text{where} \quad dr/dt = u-c \tag{307}$$

The value of u is smaller than its maximum value, $u_2 \doteq (c_1/\gamma)(\xi-1)$, and dr/dt is very nearly equal to $-c_1$. Therefore

$$\frac{dQ}{dr} < \frac{2c_1}{\gamma}(\xi-1)\,\frac{1}{r} \tag{308}$$

or

$$Q < \frac{2c_1}{\gamma}\,(\xi-1)\,\ln(r/R_{Blast}) \tag{309}$$

and the value of Q is negligible with respect to c. Therefore taking

$$Q = 0, \quad u = \sigma\ , \text{ and } P = u + \sigma = \frac{4}{\gamma-1}\,(c - c_1) \tag{310}$$

But

$$dP/dt = -2uc/r = -4c(c-c_1)/(\gamma-1)r \tag{311}$$

when:

$$dr/dt = u + c = \left[2/(\gamma-1)\right](c-c_1) + c \tag{312}$$

Differentiating P from Eqs. (310) and substituting into (311):

[redacted]

$$dc/dt = -\, c(c-c_1)/r \tag{313}$$

Combining Eqs. (312) and (313),

$$\frac{dc}{dr} = -\,\frac{1}{r}\;\frac{c(c-c_1)}{c + \left[2/(\gamma-1)\right](c-c_1)} \tag{314}$$

or

$$\frac{dr}{r} = -\, dc \left[\frac{1}{c-c_1} + \left(\frac{2}{\gamma-1}\right)\frac{1}{c}\right] \tag{315}$$

and integrating Eq. (315),

$$r \; c^{2/(\gamma-1)} \; (c-c_1) = G \tag{316}$$

Here G is a constant along the characteristic.

If we know that at the point $r = r_o$ at the time $t = t_o$ the velocity of sound has the value $c = c_o$ and from Eq. (316) the corresponding value of the constant is $G = G_o$, then we can use Eqs. (313) and (316) to tell us the corresponding values of r and c at a subsequent time, t. Using Eq. (316) to eliminate r in Eq. (313):

$$\frac{dc}{dt} = -\,\frac{1}{G_o}\; c^{(\gamma+1)/(r-1)}\; (c-c_1)^2 \tag{317}$$

or integrating

$$t - t_o = -\, G_o \int_{c_o}^{c} \frac{dc}{c^{(\gamma+1)/\gamma-1)}\;(c-c_1)^2} \tag{318}$$

If $(\gamma + 1)/(\gamma - 1)$ is an integer, the integration may be carried out explicitly. Thus (using Dwight's Tables of Integrals 161.2),

[redacted]

For $\gamma = 3$

$$t - t_0 = \frac{G_0}{c_1^2}\left[\left(\frac{1}{c} - \frac{1}{c_0}\right) + \left(\frac{1}{c-c_1}\right) - \left(\frac{1}{c_0-c_1}\right) + \frac{2}{c_1}\,\ln\,\frac{c_0(c-c_1)}{c(c_0-c_1)}\right] \qquad (319)$$

and for other values of γ such that $(\gamma + 1)/(\gamma - 1)$ is an integer, the explicit integration is carried out by means of a simple recursion relation (Dwight's Tables of Integrals 161.29). From Eq. (318) or (319) we can tell the value of c for any time and then from Eq. (316) we know the corresponding radius.

There is only one objection to the above procedure. The front of the blast waves does not travel as fast as the propagation of the characteristics. Therefore the position of the shock front must be calculated separately and any values of the radius obtained by the above procedure which lie in front of the shock front must be discarded. At any time, the position of the shock wave, R, may be determined from the integral:

$$R = \int U\,dt + \text{constant} \qquad (320)$$

Here the integration must be carried out numerically with the help of Eq. (296):

$$U = c_1 + \left(\frac{\gamma + 1}{4\gamma}\right) c_1\,(\xi - 1) + \ldots \qquad (296')$$

The value of ξ to use in (296') can be computed from the values of c obtained from the characteristics.

The above treatment is particularly useful in analyzing the results of a blast meter which measures the pressure at a given point as a function of time. Knowing the pressure for all times at this one point, we can then calculate the values of G and obtain the shape of the blast wave and the pressure at any other position.

[redacted]

[redacted]

(23). TAYLOR'S TREATMENT OF STRONG BLAST WAVES (SIMILARITY SOLUTION).

Taylor developed a similarity solution for the conditions within and behind strong blast waves. He treats the radius of the blast wave, R, as an independent variable. All of the properties of the gas behind the blast wave are then expressed in terms[20] of R and $y = r/R$. Assuming that the blast wave starts expanding from a point source, Taylor seeks solutions to the equation of motion, equation of continuity, and the equation for the conservation of entropy (behind the blast) such that:

$$p/p_1 = A^2 R^{-3} f(y)/c_1^2 \tag{321}$$

$$\rho/\rho_1 = g(y) \tag{322}$$

$$u = A R^{-3/2} h(y) \tag{323}$$

$$dR/dt = A R^{-3/2} \tag{324}$$

Here the constant, A, is related to the energy of the system. In replacing the variables r and t in the hydrodynamical equations by R and y it is convenient to set:

$$\left(\frac{\partial}{\partial t}\right)_r = \frac{dR}{dt}\frac{\partial}{\partial R} - \frac{y}{R}\frac{dR}{dt}\frac{\partial}{\partial y} = AR^{-3/2}\left[\frac{\partial}{\partial R} - \frac{y}{R}\frac{\partial}{\partial y}\right] \tag{325}$$

$$\left(\frac{\partial}{\partial r}\right)_t = \frac{1}{R}\frac{\partial}{\partial y} \tag{326}$$

The equation of motion (253) then becomes:

$$(3/2)\, h - y h' + hh' + f'/(\gamma g) = 0 \tag{327}$$

and the equation of continuity (254) becomes:

20) We have changed Taylor's notation in the following way:

Taylor notation	η	ψ	f_1	ϕ
Notation this report	y	g	f	h

[redacted]

$$(h-y)\, g' + 2\, g\, h/y + h'\, g = 0 \tag{328}$$

Since the equation for the conservation of entropy for an element of fluid (after passing through the blast wave) following this element of fluid in its motion is

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial r}\right)\,(p\rho^{-\gamma}) = 0 \tag{329}$$

it follows that:

$$f' = \frac{f\left[-3y + (3 + \tfrac{1}{2}\gamma)h - 2\gamma h^2/y\right]}{\left[(y-h)^2 - f/g\right]} \tag{330a}$$

$$h' = \left[(1/\gamma)\,(f'/g) - 3h/2\right] / (y-h) \tag{330b}$$

$$g' = g\left[h' + 2h/y\right] / (y-h) \tag{330c}$$

These equations can be solved for f', g', and h' in terms of f, g, and h. Knowing the values of the functions f, g, and h, at the shock wave, i.e. $y=1$, we can integrate these equations numerically to determine their values at any other value of y. At the shock front:

$$U = dr/dt = A\, R^{-3/2} \tag{331}$$

$$g(1) = \rho_2/\rho_1 = (\gamma+1)/(\gamma-1) \tag{332}$$

$$h(1) = u_2/U = 2/(\gamma+1) \tag{333}$$

$$f(1) = (c_1^2/U^2)\,(p_2/p_1) = 2\gamma/(\gamma+1) \tag{334}$$

Solving these equations numerically for air with $\gamma = 1.4$ and for a substance with $\gamma = 5/3$ Taylor[21] obtained the following results shown in Table I and Fig. 26.

---

21) G. I. Taylor (BM-35; RC-210) p. 12

[redacted]

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TABLE I

Air with $\gamma = 1.4$

y	f	h	g
1.00	1.167	0.833	6.000
0.98	0.949	0.798	4.000
0.96	0.808	0.767	2.808
0.94	0.711	0.737	2.052
0.92	0.643	0.711	1.534
0.90	0.593	0.687	1.177
0.88	0.556	0.665	0.919
0.86	0.528	0.644	0.727
0.84	0.507	0.625	0.578
0.82	0.491	0.607	0.462
0.80	0.478	0.590	0.370
0.78	0.468	0.573	0.297
0.76	0.461	0.557	0.239
0.74	0.455	0.542	0.191
0.72	0.450	0.527	0.152
0.70	0.447	0.513	0.120
0.68	0.444	0.498	0.095
0.66	0.442	0.484	0.074
0.64	0.440	0.470	0.058
0.62	0.439	0.456	0.044
0.60	0.438	0.443	0.034
0.58	0.438	0.428	0.026
0.56	0.437	0.415	0.019
0.54	0.437	0.402	0.014
0.52	0.437	0.389	0.010
0.50	0.436	0.375	0.007

Approximate Calculation for $\gamma = 5/3$

y	f	h	g
1.00	1.250	0.750	4.00
0.95	0.892	0.680	2.30
0.90	0.694	0.620	1.14
0.80	0.519	0.519	0.63
0.70	0.425	0.445	0.29
0.50	0.379	0.300	0.05
0.00	0.344	0.000	0.00

[redacted]

[redacted]

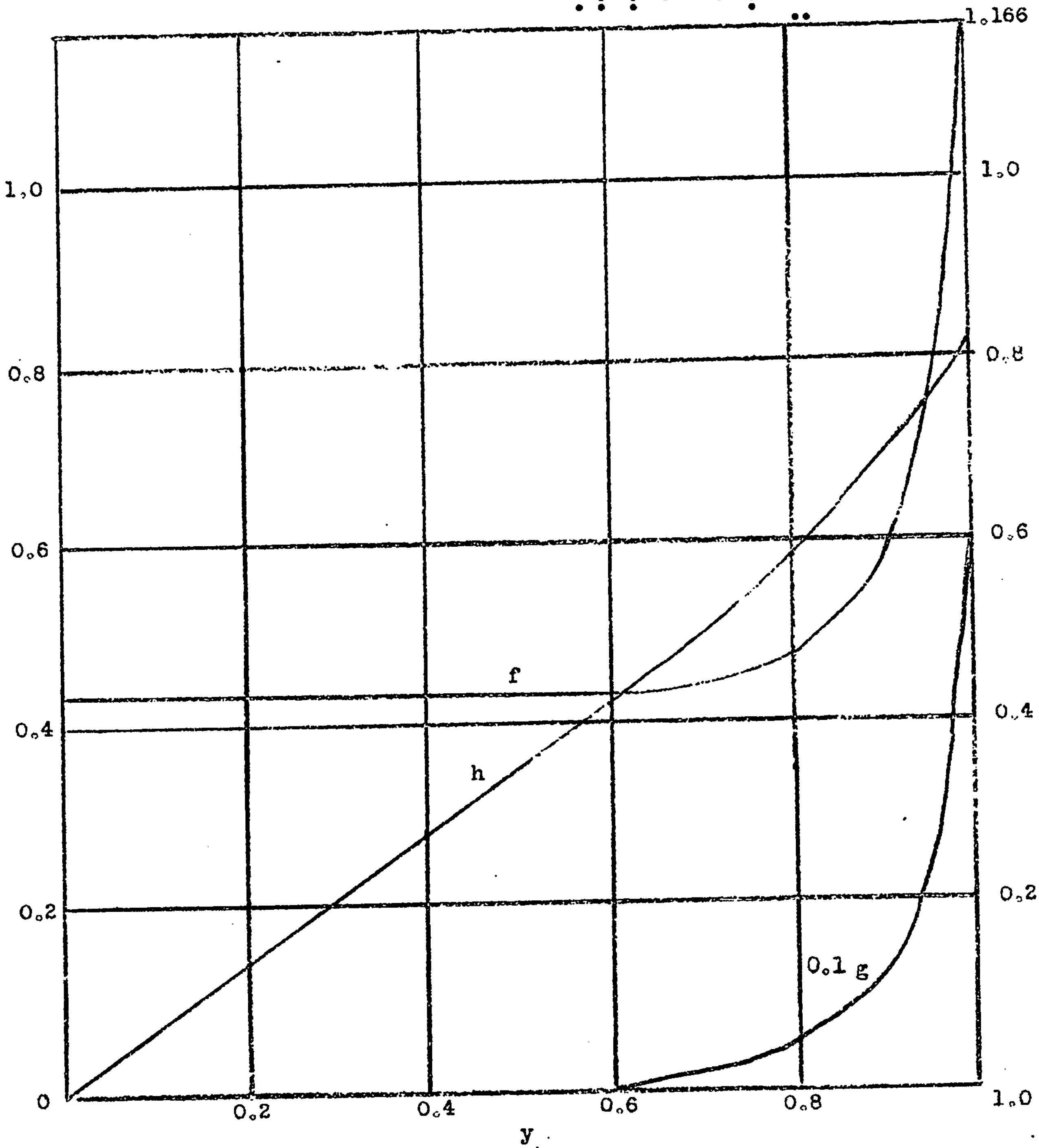

Figure 26, Air, with $\gamma = 1.4$.

Unfortunately, we cannot satisfy the shock-wave boundary conditions for weak shock waves. Therefore this solution is only satisfactory for strong blasts and becomes progressively less satisfactory as the blast becomes weak.

[redacted]

[redacted]

It is easy to obtain the total energy behind the blast wave in terms of f, g, and h. Let this energy be E_{tot}, then:

$$E_{tot} = 4\pi \int_0^R \left(\tfrac{1}{2} \rho u^2 + p/(\gamma-1) \right) r^2 dr$$

$$= 4\pi \int_0^1 \left[\tfrac{1}{2} \rho_1 g A^2 R^{-3} h^2 + \left(\frac{1}{\gamma-1} \right) p_1 \frac{A^2 R^{-3} f}{c_1^2} \right] R^3 y^2 dy$$

$$= 4\pi\rho_1 A^2 \int_0^1 \left[\tfrac{1}{2} g h^2 + \frac{f}{\gamma(\gamma-1)} \right] y^2 dy \qquad (335)$$

The total energy is therefore expressed in terms of a definite integral which is only a function of gamma. Thus for air with $\gamma = 1.4$, we get:

$$E_{tot} = 5.36\ \rho_1 A^2 \qquad (336)$$

and using this expression to eliminate A^2:

$$p = .133\ E_{tot}\ f/R^3 \qquad (337)$$

$$u = .442\ (E_{tot}/\rho_1)^{1/2} R^{-3/2} h \qquad (338)$$

$$U = .442\ (E_{tot}/\rho_1)^{1/2} R^{-3/2} \qquad (339)$$

$$\rho = \rho_1 g \qquad (340)$$

These equations form a complete solution to the strong-blast-wave problem. Notice that from these equations it is clear that, for a given total energy, p is independent of the atmospheric pressure or density, u and U are inversely proportional to the square root of the atmospheric pressure or density, and the time scale is proportional to the square root of the atmospheric pressure or density.

The energy W dissipated (in heating the air behind the shock wave) can be expressed in terms of the Taylor functions since (using Eq. 72):

[redacted]

$$W = 4\pi T_1 C_p \rho_1 \int_0^r \ln\left[(p/p_1)^{1/\gamma} (\rho_1/\rho)\right] r^2 \rho \, dr \qquad (341)$$

To get a lower limit to the dissipation, this integral can be carried out to a distance, R_o, where according to the Taylor equations the pressure in the shock front is reduced to the initial pressure. Because of the poorness of the approximations involved when the shock wave goes from strong to weak, this does not give a very accurate value. Thus Penney (BM-37; RC-260) found for air:

W/E_{tot}	.18	.33	.52	.61	.64	.64
R/R_o	.1	.2	.4	.6	.8	1.0

(see also W. G. Penney and K. J. Kynch; BM-47: RC-286).

The energy dissipation from a point-source explosion according to the Taylor theory is not very accurate since the rate of dissipation of energy is still appreciable when the overpressure is a few atmospheres. This is true because the large area of the shock front at the lower pressures nearly compensates for the much lower dissipation per unit area of the shock front. Unfortunately the similarity solution of the point-source explosion is not valid as far as this. Nevertheless numerical integrations have succeeded in evaluating the blast wave to such a radius that the overpressure is nearly as small as at the limiting radius of military importance. The total dissipation at the stage where the overpressure is of the order of one atmosphere is about 80 percent of the energy release. Such an estimate of course only applies to the highly idealized system envisaged by Taylor.

Other numerical estimates of the energy dissipation in the blast wave from an explosion can be made. Experiments in air on bare charges have succeeded in contributing contours of the shock front at various times. Numerical integration over the shock front at various times have shown that the energy dissipation up to the stage at which the overpressure is a few pounds to the square inch is roughly equal to the usually accepted value of the chemical energy

[redacted]

[redacted]

of the charge. Since the blast wave at this stage still has an energy content of about one quarter of the chemical energy, there is an apparent discrepancy in the energy balance. The most likely explanation is that the extra energy results from afterburning of the products of the explosion at the early stages when the interface between the explosive products and the air is not sharply defined, because a sharp interface would be unstable.

The energy dissipation in water can also be calculated using purely theoretical results on the shock-wave pressures near in. Roughly thirty percent of the chemical energy is wasted irreversibly in heat by the stage that the shock pressure is of the order of one ton per square inch, i.e., at approximately 50 charge radii.

(24) VON NEUMANN THEORY OF BLAST WAVES[22] (GREATER GENERALITY BUT STILL WITH SIMILARITY).

Von Neumann has developed a theory of blast waves which is slightly more general than the treatment of either Taylor or Penney since it is applicable to one-, two-, or three-dimensional problems and can be used with general boundary conditions. For example, it would not be necessary to have constant density outside of the blast wave. However, to illustrate the method let us confine our discussion to the same spherical expansion problem treated by G. I. Taylor (see last section). For this purpose, let us define:

R = blast wave radius

r_0 = co-ordinate of particle at time $t = 0$

r = co-ordinate of particle at time $t = t$

J = ratio of kinetic to internal energy of particles = $\frac{1}{2}\ u^2/E$

Von Neumann then seeks a solution satisfying the similarity conditions:

$$r_0/R = Z(J) \tag{342}$$

$$r/R = y(J) \tag{343}$$

22) G. J. Kynch, (BM-82; MS-69). θ in the Kynch treatment is called J in the above.

[redacted]

[redacted]

He then takes t and J as his independent variables. Thus:

$$dr_o = Z\,\frac{dR}{dt}\,dt + RZ'\,dJ \tag{344}$$

$$dr = y\,\frac{dR}{dt}\,dt + Ry'\,dJ \tag{345}$$

or eliminating dJ between these equations,

$$dr = \frac{dR}{dt}\left[y - \frac{Zy'}{Z'}\right]dt + \frac{y'}{Z'}\,dr_o \tag{346}$$

Therefore:

$$u = (\partial r/\partial t)_{r_o} = (dR/dt)\left[y - Zy'/Z'\right] \tag{347}$$

and

$$(\partial r/\partial r_o)_t = y'/Z' \tag{348}$$

For strong shock waves, we have the boundary conditions at the wave front:

$$J = 1 \qquad p/\rho_1 = p_2/\rho_1 = \frac{2}{1+\gamma}\left(\frac{dR}{dt}\right)^2$$

$$u = u_2 = \frac{2}{1+\gamma}\,\frac{dR}{dt} \qquad E = E_2 = \frac{1}{2}\,u_2^2 = \frac{2}{(1+\gamma)^2}\left(\frac{dR}{dt}\right)^2$$

$$\rho/\rho_1 = \eta = (\gamma+1)/(\gamma-1)$$

In addition to satisfying the boundary conditions, the functions y and Z must satisfy the four equations:

1) Equation of continuity.

$$\rho/\rho_1 = \left(\frac{\partial r_o^3}{\partial r^3}\right)_t = \frac{Z^2 Z'}{y^2 y'} \tag{350}$$

2) Equation of adiabatic motion (after shock wave has passed).

$$p/p_2 = (\rho/\rho_2)^\gamma \tag{351}$$

3) Equation of state

$$E = p/\left[\rho(\gamma-1)\right] \tag{352}$$

[redacted]

[redacted]

4) Equation of conservation of energy. [This together with the equation for the adiabatic motion is equivalent to the equation of motion]. We will develop this condition later. First, use the above equations together with the boundary conditions to express E in terms of Z and y.

$$E = \frac{p}{\rho(\gamma-1)} = \frac{p_2 \rho^{\gamma-1}}{(\gamma-1)\rho_2^{\gamma}} = \frac{p_2 \rho_1^{\gamma-1}}{(\gamma-1)\rho_2^{\gamma}} \left(\frac{Z^2 Z'}{y^2 y'}\right)^{\gamma-1} \tag{353}$$

$$= \left(\frac{1}{\gamma-1}\right)\left(\frac{2}{1+\gamma}\right)\left(\frac{dR}{dt}\right)^2 \left(\frac{Z^2 Z'}{y^2 y'}\right)^{\gamma-1} \left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}$$

$$= \left(\frac{2}{\gamma^2-1}\right)\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} \left(\frac{Z^2 Z'}{y^2 y'}\right)^{\gamma-1} \left(\frac{dR}{dt}\right)^2$$

Then

$$J = \frac{u^2/2}{E} = \frac{(y - Zy'/Z')^2/2}{\left(\frac{2}{\gamma^2-1}\right)\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\left(\frac{Z^2 Z'}{y^2 y'}\right)^{\gamma-1}} \tag{354}$$

This equation gives one relationship between J and f, Z, f', Z'. Now to get the total energy in the system, E_{tot}, we perform the integration:

$$E_{tot} = 4\pi \int_0^R \rho(E + u^2/2)\, r^2\, dr \tag{355}$$

But

$$dJ = dr/Ry' \tag{256}$$

$$\rho = \rho_1 \; Z^2 Z'/y^2 y'$$

$$r^2 = R^2 y^2$$

$$E + u^2/2 = E\,(1 + J)$$

Therefore

$$E_{tot} = 4\pi \int_{J_0}^{1} \rho_1\, E(1+J)\, R^3 Z^2 Z'\, dJ \tag{357}$$

[redacted]

[redacted]

Let

$$G(J) = \left(\frac{2}{\gamma^2-1}\right)\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\left(\frac{Z^2Z'}{y^2y'}\right)^{\gamma-1} Z^2Z'\,(1+J)$$

$$= (1/2)\left(\frac{1}{J}+1\right) Z^2Z'\left(y - \frac{Zy'}{Z'}\right)^2 \qquad (358)$$

Then

$$E_{tot} = 4\pi\int_{J_o}^{1} \rho_1\, R^3(dR/dt)^2\, G(J)\, dJ$$

$$= R^3(dR/dt)^2\, 4\pi\int_{J_o}^{1} \rho_1\, G(J)dJ \qquad (359)$$

If the gas extends to the center, J_o is zero. The equation of conservation of energy states that E_{tot} is constant with respect to time. The only way this is possible is for

$$R^3(dR/dt)^2 = \text{constant} \qquad (360)$$

or

$$dR/dt = A\, R^{-3/2} \qquad (361)$$

This result is then the same as in Taylor's theory.

The total energy of the gas lying within a small sphere whose radius is determined by the condition that the ratio of its kinetic to its potential energy is J is given by the equation:

$$E_{tot}\,(J) = 4\pi R^3(dR/dt)^2\int_{J_o}^{J} \rho_1\, G(J)\, dJ \qquad (362)$$

The rate at which this gas does work on the surrounding gas is given by:

$$4\pi r^2 pu = -\left(\frac{\partial E_{tot}(J)}{\partial t}\right)_{r_o} = -\frac{d\,E_{tot}(J)}{dJ}\left(\frac{\partial J}{\partial t}\right)_{r_o} \qquad (363)$$

[redacted]

[redacted]

But
$$\left(\frac{\partial J}{\partial t}\right)_{r_0} = -\frac{Z}{Z'R}\frac{dR}{dt} \tag{364}$$

$$\frac{dE_{tot}(J)}{dJ} = 4\pi R^3\rho_1\, G(J) = 4\pi R^3\rho_1 \left(\frac{2}{\gamma^2-1}\right)\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} \cdot$$
$$\left(\frac{Z^2Z'}{y^2y'}\right)^{\gamma-1} Z^2Z'(1+J) \tag{365}$$

$$r^2 = R^2y^2 \tag{366}$$

$$p = p_2\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\left(\frac{Z^2Z'}{y^2y'}\right)^{\gamma} = \frac{2}{\gamma+1}\rho_1\left(\frac{dR}{dt}\right)^2\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\left(\frac{Z^2Z'}{y^2y'}\right)^{\gamma} \tag{367}$$

$$u = \frac{dR}{dt}\left(y - \frac{Zy'}{Z'}\right) \tag{368}$$

So that using Equations (353) through (368), cancelling and rearranging Eq.(365) becomes:

$$\frac{yZ'}{Zy'} = \frac{\gamma + J}{\gamma - 1} \tag{369}$$

And substituting this into the equation for the conservation of energy:

$$\frac{1}{y^2}\left(\frac{Z}{y}\right)^{3(\gamma-1)} = \frac{\frac{1}{2}\left(\frac{J+1}{\gamma+J}\right)^2}{\left(\frac{2}{\gamma^2-1}\right)\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\left(\frac{\gamma+J}{\gamma-1}\right)^{\gamma-1} J} \tag{370}$$

Solving these two equations simultaneously, von Neumann obtained y and Z in the completely analytical form:

$$Z(J) = J^{\gamma/(2\gamma+1)}\left(\frac{J+1}{2}\right)^{2/5}\left(\frac{3(2-\gamma)J+2\gamma+1}{7-\gamma}\right)^{-\frac{13\gamma^2+7\gamma-12}{5(2-\gamma)(2\gamma+1)}} \tag{371}$$

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$$y(\xi) = J^{\gamma/(2\gamma+1)} \left(\frac{J+1}{2}\right)^{-2/5} \left(\frac{3(2-\gamma)J+2\gamma+1}{7-\gamma}\right)^{-\frac{13\gamma^2-7\gamma+12}{5(2\gamma+1)(3\gamma-1)}} \left(\frac{J+\gamma}{1+\gamma}\right)^{(\gamma+1)/(3\gamma-1)} \quad (372)$$

At the origin, J is zero.

In case the problem is q dimensional instead of three dimensional, the equation of continuity and the element of volume in the integrals are changed. Otherwise one method remains unchanged. Kynch has used this method to consider the effect of explosions in a medium of varying density.

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(25). BETHE'S MODIFICATION OF W.K.B. METHOD (WEAK SHOCKS, NO SIMILARITY)

Lecture by Bethe

Bethe developed a semiacoustical method for treating weak shocks where no similarity conditions are possible. His method is similar to the well-known W.K.B. method of quantum mechanics. It is based on acoustical theory as the zeroeth approximation.

In acoustical theory, the overpressure is made up of a wave traveling outwards and a wave traveling inwards. Thus:

$$p - p_1 = f'(t - r/c)/r + g'(t + r/c)/r \tag{373}$$

The factor $1/r$ is due to the geometrical attenuation of the pressure. Here f' and g' are arbitrary functions. The wave f' is traveling outwards since its argument remains constant when r/c increases at the same rate as t. Similarly g' represents an incoming wave. Eq. (373) in the most general spherically symmetrical solution of the acoustical equation:

$$\Delta p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \tag{374}$$

The material velocity , u, is given by the relation:

$$u = (f' - g')/\rho r c + (f + g)/\rho r^2 \tag{375}$$

Notice that the inverse square terms in the second bracket exist even for incompressible materials with infinite velocity of sound. In the acoustical theory, waves always retain their shape since the small variation in the velocity of sound for the infinitesimal pressure differences considered is neglected. This is not true in any actual case, even for very weak shocks.

The Riemann method could be used for very weak shocks in one dimension, but the additional term, $2u\rho/r$, in the equation of continuity makes it

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impossible to use it for the case of any sort of spherical wave.

Bethe has developed the W.K.B. approximation to treat those cases where the half wave length, L , is very small compared with the distance, r, from the wave to the origin. Under these conditions, $2u/r$ will be small compared with $\partial u/\partial r$, and generally, in the disturbance, the hydrodynamical variables change rapidly compared to r. With these assumptions, Bethe showed that weak wavelets which propagate with the velocity $c + u$ maintain constant values of the characteristic, $(c + u)\,r$. In q dimensions they would have the characteristic $(\sigma + u)\, r^{(q-1)/2}$. As in Eq. (305), we have:

$$\sigma = \left[2/(\gamma-1)\right](c-c_1) \tag{376}$$

Ahead of the shock wave, $\sigma - u = 0$, since both σ and u are zero. Directly behind the shock wave according to Eq. (306), $\sigma - u = 0$ through terms of the order of $(\xi-1)^2$. Farther behind the shock, $(\sigma-u)/\sigma$ becomes of the order of L/r due to the influence of a term similar to the last term in Eq. (375). If terms of the order of L/r are neglected we can assume (as in Penney's treatment) that everywhere:

$$\sigma = u \tag{377}$$

The time, t , for a signal traveling with the velocity $u + c$ to go from R_1 to R is then given by the relation:

$$t = \int_{R_1}^{R} \frac{dr}{u+c} = \int_{R_1}^{R} \frac{dr}{c_1 + \left[(\gamma-1)/2\right]\sigma + u} = \int_{R_1}^{R} \frac{dr}{c_1 + \left[(\gamma+1)/2\right]u} \tag{378}$$

Now at sufficiently large distances from the origin where the inverse square terms have become negligible in the acoustical case, the veloctiy of a given wavelet decays in the following manner:

$$u(r) = \omega r^{-(q-1)/2} \tag{379}$$

Here ω is a constant characteristic of the wavelet and q is the number of dimensions of the problem under consideration ($q = 1$ for a plane wave, 2 for a cylindrical wave, and 3 for a spherical wave). Remembering that u is small compared to c_1 :

$$t = \int_{R_1}^{R} \frac{dr}{c_1 + \left[(\gamma+1)/2\right] \omega r^{-(q-1)/2}} = \frac{R-R_1}{c_1} - \left(\frac{\gamma+1}{2c_1^2}\right)\omega \int_{R_1}^{R} r^{-(q-1)/2}\, dr \tag{380}$$

$$= \frac{R-R_1}{c_1} - \frac{(\gamma+1)\omega}{c_1^2(3-q)} \left[R^{(3-q)/2} - R_1^{(3-q)/2}\right] \quad \text{(Except for } q = 3\text{)}$$

$$= \frac{R-R_1}{c_1} - \frac{(\gamma + 1)\omega}{2c_1^2} \ln\,(R/R_1) \quad \text{(for } q = 3\text{)}$$

Formation of Shocks

Thus if two parts of a wavelet initially have different mass velocities, they will travel at different velocities, (namely faster when σ and u are larger). This makes the compression phase of the wavelet become steeper and the rarefaction part become more extended. These effects will be more pronounced in the one-dimensional than in the two-or three-dimensional cases. The following example will make this clear.

Suppose that at the time $t = 0$, we have a sinusoidal pulse traveling outwards. This is shown in Fig. 27(a). At a somewhat later time, t, this wave has assumed the shape shown in Fig. 27(b) and after a sufficiently long time it assumes the limiting form shown in Fig. 27(c). These drawing would have similar shapes if we plotted the velocity of sound, c, versus position rather than pressure versus position. If the amplitude of the waves is small and they have travelled sufficiently far so that the inverse square contribution to the material velocity can be neglected, then from Eqs. (373) and (375) it follows that:

$$u = (p-p_1)/\rho c \doteq (p-p_1)/\rho_1 c_1 \tag{381}$$

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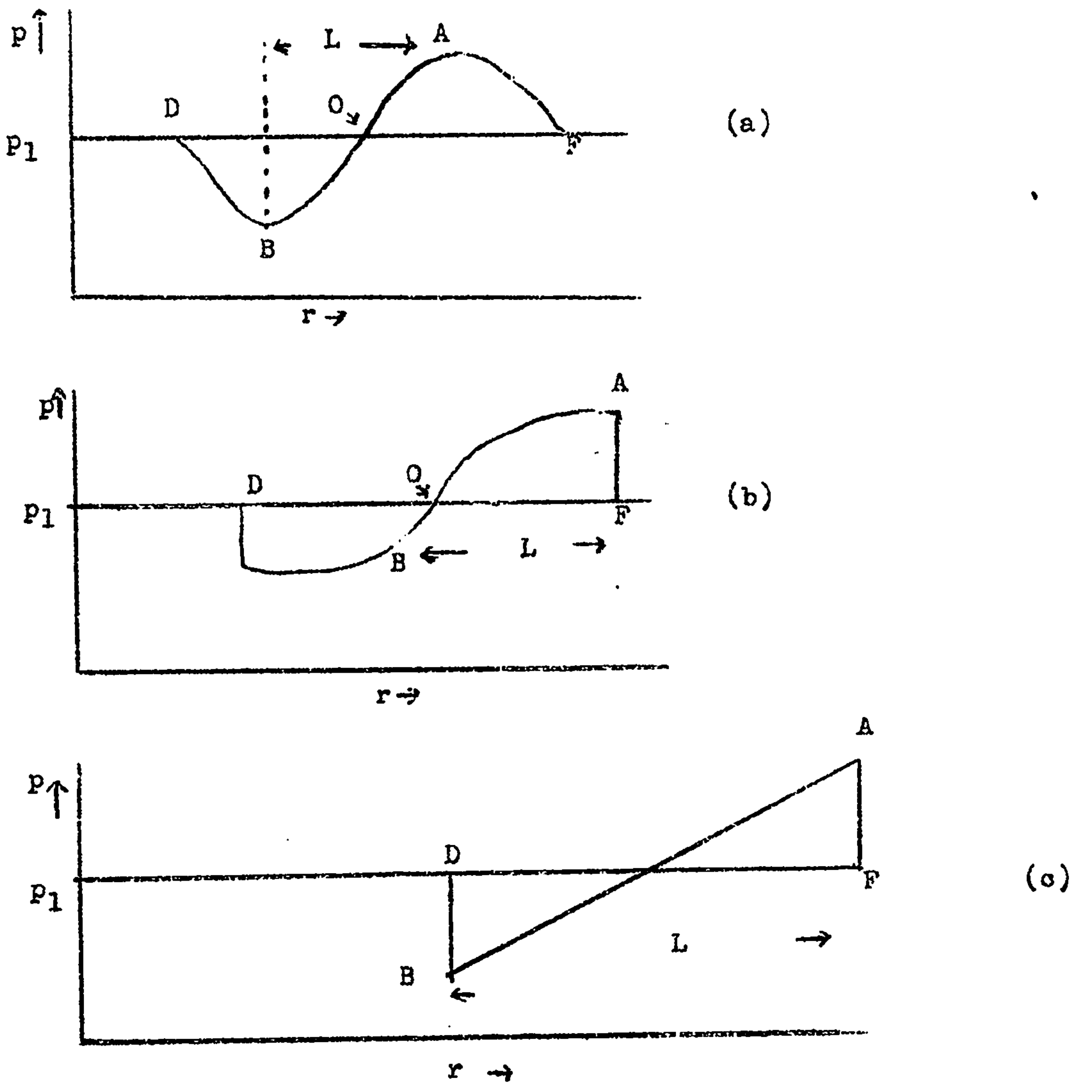

Figure 27

Thus the greater the pressure the greater the material velocity.

If the overpressure at A is initially $p_m = p_A - p_1$ and if the position of A at this time is R_{A1}, then:

$$\frac{\omega_A}{c_1} = \frac{p_m}{\rho_1 c_1^2} R_{A1}^{(q-1)/2} = \frac{1}{\gamma} \frac{p_m}{p_1} R_{A1}^{(q-1)/2} \qquad (382)$$

 [redacted]

Since the wave length is supposed to be small compared to the distance R_{A1}, it follows that $\omega_B/c_1 = -\omega_A/c_1$. At the points F and D, $\omega = 0$. Now consider the implications of Eq. (380) for the behavior of waves on different numbers of dimensions.

After the time t, the point A has moved the distance $R_A - R_{A1}$ given by the equation:

$$c_1 t = R_A - R_{A1} - \left(\frac{\gamma + 1}{3 - q}\right) \frac{1}{\gamma} \frac{p_m}{p_1} R_{A1}^{(q-1)/2} \left[R_A^{(3-q)/2} - R_{A1}^{(3-q)/2}\right]$$

(except for q = 3) (383)

$$= R_A - R_{A1} - \left(\frac{\gamma+1}{2\gamma}\right) \frac{p_m}{p_1} R_{A1} \ln R_A/R_{A1} \qquad \text{(for } q = 3\text{)}$$

In this time, the point F has moved the distance $R_F - R_{F1}$ given by the equation:

$$c_1 t = R_F - R_{F1} \tag{384}$$

At the time, t_{AF}, when A overtakes F:

$$(R_A - R_{A1}) - (R_F - R_{F1}) = \frac{L}{2} = \left(\frac{\gamma+1}{3-q}\right) \frac{1}{\gamma} \frac{p_m}{p_1} R_{A1}^{(q-1)/2} \left[R_A^{(3-q)/2} - R_{A1}^{(3-q)/2}\right]$$

(except for q = 3)

$$= \left(\frac{\gamma+1}{2\gamma}\right) \frac{p_m}{p_1} R_{A1} \ln (R_A/R_{A1}) \qquad \text{(for } q = 3\text{)}$$

[redacted]

 [redacted]

Thus when q = 1

$$c_1 t_{AF} = \left(\frac{\gamma}{\gamma+1}\right) L \frac{p_1}{p_m} - \left\langle \frac{L}{2} \right\rangle \qquad (385)$$

When q = 2

$$c_1 t_{AF} = \left(\frac{\gamma}{\gamma+1}\right) L \frac{p_1}{p_m} - \left\langle \frac{L}{2} \right\rangle + \left(\frac{\gamma}{\gamma+1}\right)^2 \frac{L^2}{4R_{A1}} \left(\frac{p_1}{p_m}\right)^2 \qquad (386)$$

When q = 3

$$c_1 t_{AF} = -\left\langle \frac{L}{2} \right\rangle + R_{A1} \, e^{\left[\gamma/(\gamma+1)\right] (p_1/p_m) L/R_{A1}} - R_{A1} \qquad (387)$$

When q = 4

$$c_1 t_{Ac} = -\left\langle \frac{L}{2} \right\rangle + \frac{\left(\frac{\gamma}{\gamma+1}\right) L \frac{p_1}{p_m} - \left(\frac{\gamma}{\gamma+1}\right)^2 \frac{L^2}{4R_{A1}} \left(\frac{p_1}{p_m}\right)^2}{1 - \left(\frac{\gamma}{\gamma+1}\right) \frac{L}{R_{A1}} \frac{p_1}{p_m} + \left(\frac{\gamma}{\gamma+1}\right)^2 \left(\frac{L^2}{4R_{A1}^2}\right) \left(\frac{p_1}{p_m}\right)^2} \qquad (388)$$

Here the terms $\langle L/2 \rangle$ are neglected since the time for the shock to be formed, t_{AF}, is considered to be long compared to the time for a signal with the velocity of sound to travel across the wave. This approximation is inherent throughout the theory. Better results would not be obtained by the inclusion of these terms because of compensating errors which will be explained later. Thus it takes progressively longer time for a shock wave to develop in one, two, or three dimensions. In more than three dimensions shock waves only occur if

$$\left[\gamma/(\gamma + 1)\right] \quad (L/R_{A1}) \quad (p_1/p_m)$$

is small compared to unity.

Similar developments could be carried out for the time required for D to overtake B. The time for this second shock to develop is approximately the same as the time for the front shock to develop. After the shocks are formed,

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the linearity of u with overpressure insures that the pressure in the pulse will become linear with distance as shown in Fig. 27(c). Since the front shock moves with the velocity characteristic of the peak pressure and the second shock moves at a slower velocity characteristic of the initial pressure, the two shocks will separate and the wave will spread.

In the case of a periodic wave, the wave length remains invariant. However if the wave is originally sinusoidal, it will become saw-toothed with the peaks corresponding to the original positions of maximum pressure. The waves cannot spread because the shocks have the same pressures and therefore travel at the same velocity.

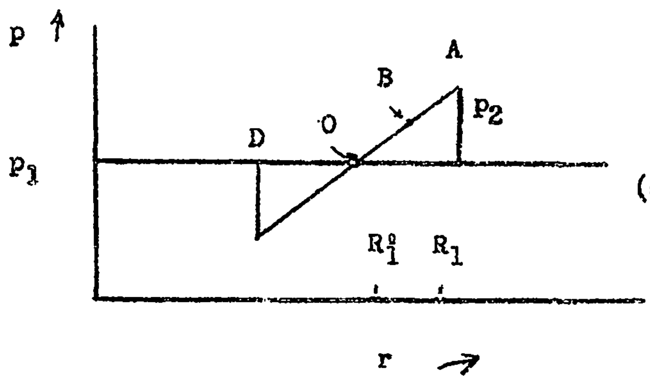
Decay of Shock Waves

Next we can consider the decay of shock waves from two different standpoints. The first makes use of this semiacoustical method and the second makes use of the thermodynamical arguments stated previously. Both lead to the same results. Consider a shock wave as shown in Fig. 27(a). If the peak pressure is p_2, the velocity of the shock front is given by Eq. (296):

$$U^2 = c_1^2 + \frac{\gamma + 1}{2\gamma}\left(\frac{p_2 - p_1}{p_1}\right) \qquad (389)$$

or expanding:

$$U = c_1\left[1 + \frac{\gamma + 1}{4\gamma}\left(\frac{p_2 - p_1}{p_1}\right)\right] = c_1 + \left(\frac{\gamma+1}{4}\right) u_2 \qquad (390)$$



(a)

$$= c_1 + \left(\frac{\gamma+1}{4}\right) \omega r^{-(q-1)/2}$$

Here ω is a property of a wavelet which may be superposed on the main pulse at any instant (see Eq. 379).

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The time required for a shock wave to go from R_1 to R is then given by the equation:

$$t = \int_{R_1}^{R} dr/U = \int_{R_1}^{R} \frac{dr}{c_1\left[1 + \left[(\gamma+1)/4\right](\omega/c_1)r^{-(q-1)/2}\right]} = \frac{1}{c_1}(R-R_1)$$

$$- \left(\frac{\gamma+1}{4}\right)\frac{1}{c_1^2}\int_{R_1}^{R} \omega r^{-(q-1)/2}dr \qquad (391)$$

The time required for the overtaking wavelet to reach R from its initial position R_1' is:

$$t = \frac{1}{c_1}(R-R_1') - \left(\frac{\gamma+1}{3-q}\right)\frac{\omega}{c_1^2}\left(R^{(3-q)/2} - R_1'^{\,(3-q)/2}\right) \text{(except for } q = 3)$$

$$= \frac{1}{c_1}(R-R_1') - \left(\frac{\gamma+1}{2}\right)\frac{\omega}{c_1^2}\,\ln\,(R/R_1') \quad \text{(for } q = 3) \qquad (392)$$

When t is small, the overtaking wavelets come from positions R_1' close to R_1; when t is large, R is sufficiently large compared to the wave length that the difference between R_1' and R_1 is negligible. Thus we can always neglect the difference between R_1 and R_1' and equate the travel times of the shock wave and the overtaking wavelet. This shows the type of approximations which are inherent in this method. The distances between various parts of the pulse are supposed to be small compared to the distance to the center. This is usually a good approximation in the case of a blast wave from a high-explosive charge but it would not be a good approximation in the case of a slow gas explosion.

$$\int_{R_1}^{R} \omega r^{-(q-1)/2}dr = \left(\frac{4}{3-q}\right)\omega\left(R^{(3-q)/2} - R_1^{(3-q)/2}\right) \text{ (except for } q = 3)$$

$$= 2\,\omega\,\ln(R/R_1) \quad \text{(for } q = 3) \qquad (393)$$

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Taking the derivative of both sides of this equation:

$$\omega R^{-(q-1)/2} = \left(\frac{4}{3-q}\right) \frac{d}{dR}\left[\omega\left(R^{(3-q)/2} - R_1^{(3-q)/2}\right)\right] \quad \text{(except for } q = 3)$$

$$= \frac{4}{3-q}\left[\left(\frac{3-q}{2}\right)\omega R^{-(q-1)/2} + \left(R^{(3-q)/2} - R_1^{(3-q)/2}\right)\frac{d\omega}{dR}\right] \quad (394)$$

$$\omega R^{-1} = 2 \ln(R/R_1) \quad d\omega/dR - 2\omega R^{-1} \qquad \text{(for } q = 3) \qquad (395)$$

From which it follows that

$$\left[1 - (R_1/R)^{(3-q)/2}\right] \frac{d \ln \omega}{d \ln R} = -\left(\frac{3-q}{4}\right) \qquad \text{(except for } q = 3) \qquad (396)$$

$$\frac{d \ln \omega}{d \ln R} = -(1/2)\big/\left(\ln (R/R_1)\right) \quad \text{(for } q = 3)$$

So that for values of R large compared to R_1,

$$\omega \sim R^{-(3-q)/4} \qquad \text{(for } q < 3) \qquad (397)$$

$$\omega \sim \frac{1}{(\ln R)^{1/2}} \qquad \text{(for } q = 3) \qquad (398)$$

But right behind the shock front, the overpressure $p_2 - p_1$ is given by the equation (from Eqs. (296) and (297):

$$p_2 - p_1 = (\gamma p_1/c_1)\, u_2 = (\gamma p_1/c_1)\, \omega R^{-(q-1)/2} \qquad (399)$$

So that at large distances

$$(p_2 - p_1)/p_1 \sim R^{-(q+1)/4} \qquad \text{(for } q < 3) \qquad (400)$$

$$\sim 1/R(\ln R)^{1/2} \qquad \text{(for } q = 3) \qquad (401)$$

[redacted]

For $q > 3$, it is easily shown that ω at the shock front reaches a finite asymptotic value at large distances so that

$$(p_2 - p_1)/p_1 \sim R^{-(q-1)/2} \qquad \text{(for } q > 3) \tag{402}$$

The decay of the shock pressure is, of course, faster for higher number of dimensions, q, for purely geometrical reasons. However, while shock waves in more than three dimensions would decay just according to acoustic theory, they decay faster than acoustic waves for $q \leq 3$. For $q = 3$, the difference is only the slowly varying factor $\sqrt{\ln R}$. For two dimensions, the decay of a shock wave is as $R^{-3/4}$ while acoustic waves decay only as $R^{-1/2}$; for one dimension, an acoustic wave would retain its amplitude but a shock wave decays as $R^{-1/2}$.

The three dimensional shock waves produced by explosions start with little similarity to the sinusoidal wave illustrated in Fig. 27. In the region of practical interest, there is a sharp positive pulse followed by a long negative pulse. To a fair degree of accuracy, experimental results on three-dimensional shock waves in air can be represented at any time by the equation:

$$p - p_1 = (p_2 - p_1)\left[1 - (R-r)/L'\right] e^{-(R-r)/L'} \tag{403}$$

where R is the position of the front, r is the position at which the pressure is observed, and L' a half "wave length" which depends mainly on the explosive energy. Eq. (403) is empirically a much better representation of the pressure distribution than the linear relation between $p-p_1$ and r which would follow from our quasi-acoustic theory.

Relation Between Duration of Pulse and Front Pressure.

From our theory we can obtain a useful relation between the length of a shock pulse and the peak pressure (See Fig. 27). Let us consider the region

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between O and A separately from the region between D and O because the wavelet emitted from O always maintains a pressure of p_1 and no energy flows from the region behind O into the region in front of O. We shall suppose that L_o is the distance between O and A at the time $t = 0$ and L is the distance at any subsequent time. In accordance with the semi-acoustic theory, we assume the pressure, p, at any point in the shock wave to be linear with distance from the shock front, i.e., we take:

$$p - p_1 = (p_2 - p_1)\,(L - R + r)/L \tag{404}$$

In the time t, the point O moves a distance R_O given by the equation:

$$R_O = c_1\, t \tag{405}$$

And in this time the shock wave moves from R_1 to R given by the equation:

$$t = \frac{1}{c_1}\,(R - R_1) - \left(\frac{\gamma + 1}{4c_1^2}\right) \int_{R_1}^{R} \omega\, r^{-(q-1)/2}\,dr \tag{406}$$

But for very large values of R such that R_1 is negligible compared to R (cf. Eq. (393)).

$$\frac{\gamma + 1}{4c_1^2} \int_{R_1}^{R} \omega r^{-(q-1)/2}\,dr = \frac{\gamma + 1}{3 - q}\ \frac{\omega}{c_1^2}\ R^{(3-q)/2} \quad (\text{for } q < 3) \tag{407}$$

$$= \frac{\gamma + 1}{2}\ \frac{\omega}{c_1}\ \ln\,(R/R_1) \quad (\text{for } q = 3) \tag{408}$$

Therefore the length of the wave becomes

$$L = L_o + R - R_1 - R_o = L_o + \frac{\gamma + 1}{3 - q}\ \frac{\omega}{c_1}\ R^{(3-q)/2} \quad (\text{for } q < 3) \tag{409}$$

For large R, we may neglect L_o. If at the same time we express ω in terms of the front pressure from Eq. (399), we get:

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$$L = \frac{\gamma + 1}{(3-q)\gamma} \left(\frac{p_2 - p_1}{p_1}\right) R \qquad (\text{for } q < 3) \qquad (410)$$

Thus the half wave length, L, varies approximately as the distance traveled, R, times the ratio of the overpressure, $p_2 - p_1$, to the initial pressure, p_1. If we take into account the behavior of the pressure from Eq. (400), we find that the wave spreads so that:

$$L \sim R^{(3-q)/4} \qquad (\text{for } q < 3) \qquad (411)$$

For q = 3, we get corresponding to Eqs. (409) and (410):

$$L = L_o + \left(\frac{\gamma + 1}{2}\right) \frac{\omega}{c_1} \ell n(R/R_1) \sim \left(\frac{\gamma + 1}{2\gamma}\right)\left(\frac{p_2 - p_1}{p_1}\right) R \, \ell n(R/R_1) \qquad (q=3)$$

$$(412)$$

At sufficiently large values of R, taking into account the fact that the front pressure is nearly inversely proportional to R, this may be written as:

$$L = \left(\frac{\gamma + 1}{2\gamma}\right) R \left(\frac{p_2 - p_1}{p_1}\right) \ell n\left(\frac{p_1}{p_2 - p_1}\right) \qquad (q = 3) \qquad (413)$$

For air with $\gamma = 1.4$

$$L = .86 \, R\left(\frac{(p_2 - p_1)}{p_1}\right) \ell n\left[p_1/(p_2 - p_1)\right] \qquad (414)$$

For large distances we may use Eq. (401) and find:

$$L \sim \sqrt{\ell n \, R} \sim \sqrt{\ell n \left[1/(p_2 - p_1)\right]} \qquad (415)$$

Thus the three-dimensional waves spread slowly.

Experimentally, the positive impulse is frequently measured,

$$I = \int (p - p_1) \, dt \qquad (416)$$

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Here the integration is only to be taken over that part of the pulse where $p - p_1$ is positive. For the linear relation (404) between p and $R-r$, since $R - r = c_1 t$,

$$I = (p_2 - p_1)\int_0^{L/c_1}\left[1 - c_1 t/L\right] dt = (p_2 - p_1)\,L/2c_1 \tag{417}$$

The energy of the shock wave as it passes R may be written:

$$E = 4\pi R^2 \int_0^{\infty} (p - p_1)u\,dt \tag{418}$$

And since from Eq. (381):

$$u \doteq (p - p_1)/\rho_1 c_1 \tag{381}$$

It follows that

$$E = \frac{4\pi R^2}{\rho_1 c_1}\int_0^{\infty} (p - p_1)^2\,dt \tag{419}$$

So that for the linear relation (404) between p and $R-r$,

$$E = \frac{4\pi R^2}{\rho_1 c_1}(p_2-p_1)^2\int_0^{2L/c_1}\left[1 - \frac{c_1 t}{L}\right]^2 dt = \frac{2}{3}\,\frac{4\pi R^2}{\rho_1 c_1^2}(p_2 - p_1)^2 L \tag{420}$$

From the fundamental notions of energy dissipation together with our previous dimensional analysis, we can obtain another relationship between the energy, distance, and front shock pressure which does not involve the wave length or duration of the pulse. As we saw in Eq. (291), the energy dissipated when a blast wave passes through a unit volume of matter is $\rho_1\,T_1\,\Delta S$. Thus for three-dimensional waves, when the shock wave expands from R to $R + dR$, and passes through a volume of $4\pi R^2\,dR$, the energy, E, of the pulse is decreased by $4\pi R^2\,dR\,\rho_1\,T_1\,\Delta S$ or:

$$dE/dR = -4\pi R^2 \rho_1\,T_1\,\Delta S \tag{421}$$

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And since:

$$C_v T_1 = p_1/[\rho_1(\gamma - 1)] \qquad (422)$$

it follows from Eq. (73) that at the front shock:

$$T_1 \Delta S = \frac{\gamma + 1}{12\gamma^2 \rho_1} \frac{(p_2 - p_1)^2}{p_1^2} \qquad (423)$$

However, for a linear pulse there is both a front and a back shock wave of approximately the same strength. Therefore the entropy change at the back shock wave is approximately the same as at the front shock wave and so altogether the change of energy of the pulse with distance becomes:

$$\frac{dE}{dR} = - 8\pi R^2 \frac{(\gamma + 1)}{12\gamma^2} \frac{(p_2 - p_1)^3}{p_1^2} \qquad (424)$$

But from Eq. (401), for three-dimensional waves after a long time:

$$\frac{p_2 - p_1}{p_1} = \frac{\alpha}{R(\ln R)^{1/2}} \qquad (425)$$

Here α is a constant. Thus:

$$\frac{dE}{dR} = \frac{2\pi}{3} \left(\frac{\gamma + 1}{\gamma^2}\right) \frac{\alpha^3 p_1}{R(\ln R)^{3/2}} \qquad (426)$$

And integrating:

$$E = 4\pi \left(\frac{\gamma + 1}{3\gamma^2}\right) \alpha^3 p_1 (\ln R)^{-1/2} \qquad (427)$$

After eliminating α :

$$E \approx 4\pi R^3 \left(\frac{\gamma + 1}{3\gamma^2}\right) \frac{(p_2 - p_1)^3}{p_1^2} \ln R \qquad (428)$$

Equation (427) is particularly useful because it shows how the energy in the shock wave varies with distance. It indicates a slow dissipation of

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energy, a phenomenon first pointed out by Penney (BM-37, RC-260). Furthermore the constant in this equation is somewhere between one half and one third of the original energy, E_o, of the explosion. Assuming that it is one third,

$$E = (1/3)E_o \Big/ \sqrt{\ln R} \doteq (1/3)\, E_o \Big/ \sqrt{\ln\left(p_1/(p_2-p_1)\right)} \qquad (429)$$

The energy is also expressed in terms of the half wave length by Eq. (420). Solving Eqs. (420) and (428) for the half wave length:

$$L = \left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2-p_1}{p_1}\right) R \ln R \doteq \left(\frac{\gamma+1}{2\gamma}\right) R \left(\frac{p_2-p_1}{p_1}\right) \ln\left(\frac{p_1}{p_2-p_1}\right) \quad (430)$$

This relationship is the same as Eq. (413) which we obtained previously on purely kinematic considerations. This shows that in the limit of large distances, the kinemati treatment is consistent with the energy relations. However, there is considerable danger of using the energy relations for moderate pressures where the pulse has not yet reached its limiting linear form. For example, in this region of interest, the pulse has more nearly the semi-empirical form of Eq. (403) and the back shock has not yet developed. Under these conditions, the energy dissipated is one half that of the linear pulse (for the same front shock pressure) and in the relation between energy and peak pressure, Eq. (428), the numerical coefficient is one half as large.

<u>Relation Between Peak Pressure, Wave Length and Energy.</u>

In order to get a practical relation between the various quantities characterizing a shock wave originating from an explosion, it is indicated to use the semiempirical shape of the pressure pulse, Eq. (403), which is found to represent fairly well the pressure as a function of time at moderate pressures (of the order of 0.1 atm). The main characteristics of this pressure pulse are:

[redacted]

 [redacted]

1. It has a positive pressure pulse of relatively short duration and lower pressure.

2. The total impulse, $\int p\,dt$, is zero.

These two features of the pressure distribution can be made plausible. Penney has shown (BM-37, RC-260) that the impulse, $\int p\,dt$, must go to zero compared to pL'/c_1 as the wave progresses outwards. This follows simply from energy conservation. As regards the shape of the wave, the positive pulse including the shock front is formed immediately by the explosion; it is therefore originally quite short. The negative pulse is formed rather late (BM-37, RC-260) in order to fulfill condition 2 above. At the time of formation, the negative pulse involves smaller deviation from the normal pressure p_1 than the positive pulse, and this feature is preserved down to moderate pressures. Moreover, at the time of formation of the negative pulse and spatial dimensions of the shock wave are quite large so that the initial length of the negative pulse is much longer than of the positive one. The back shock wave develops only very late and is extremely small in the region of moderate pressures; it is, therefore, not taken into account at all in the wave shape Eq. (403). These arguments are meant to explain only the qualitative features of the wave shape Eq. (403); the actual analytical expression is simply a convenient way to represent a wave of these properties.

The problem now arises how to connect the wave length L' with the wave length L of the kinematic theory. L', just as L, represents the length of the positive pressure pulse. We have already pointed out that the original length, S_o, of the positive pressure pulse, is likely to be quite small. Moreover, it is easy to see that the contribution of the spreading of the wave is actually somewhat smaller than is indicated by the second term of Eq. (413). This is due to the fact that the end of the positive pressure pulse (point O in Figure 27) actually moves faster than with the unperturbed sound velocity c_1.

[redacted]

 [redacted]

The velocity of this point is $c + u$. With the pressure equal to p_1, the sound velocity $c = c_1$. However, the material velocity u is not equal to zero due to the afterflow (terms of $1/r^2$) which have been neglected in our semiacoustical theory. These afterflow terms are absent at the shock front; therefore the propagation velocity of the shock front is just as we have assumed, whereas the propagation velocity of point O is greater than assumed. This means that the second term in Eq. (413) should actually be less. We believe that it is a good approximation to cancel this correction against the initial wave length L_o, and therefore to identify L' with the value L given in Eq. (430).

If we substitute the semiempirical shape of the pulse (403) into Eq. (420) and perform the indicated integration:

$$E = \frac{\pi R^2}{\rho_1 c_1^2} (p_2 - p_1)^2 L' \tag{430}$$

But after identifying L' with L in the kinematic treatment we have by Eq. (413):

$$L' = \left(\frac{\gamma + 1}{2\gamma}\right) R \left(\frac{p_2 - p_1}{p_1}\right) \ln \left(\frac{p_1}{p_2 - p_1}\right) \tag{431}$$

Eliminating L' from Eqs. (430) and (431):

$$E = \frac{\pi R^3 p_1^2}{\rho_1 c_1^2} \left(\frac{\gamma + 1}{2\gamma}\right) \left(\frac{p_2 - p_1}{p_1}\right)^3 \ln \left(\frac{p_1}{p_2 - p_1}\right)$$

$$= \pi R^3 \left(\frac{\gamma + 1}{2\gamma^2}\right) p_1 \left(\frac{p_2 - p_1}{p_1}\right)^3 \ln \left(\frac{p_1}{p_2 - p_1}\right) \tag{432}$$

And making use of Eq.(429) for the energy:

$$R\left(\frac{p_2 - p_1}{p_1}\right) \sqrt{\ln \left(\frac{p_1}{p_2 - p_1}\right)} = \left[\frac{2\gamma^2}{3\pi(\gamma + 1) p_1}\right]^{1/3} E_o^{1/3} \tag{433}$$

[redacted]

 [redacted]

Or if R is measured in meters, $\gamma = 1.4$, E_o is in equivalent tons of TNT and p_1 is one bar (or atmosphere):

$$R \left(\frac{p_2-p_1}{p_1}\right) \sqrt{\ln\left[p_1/(p_2-p_1)\right]} = 19\, E_o^{1/3} \tag{434}$$

And from Eq. (431)

$$L' = 16\, E_o^{1/3} \sqrt{\ln\left[p_1/(p_2-p_1)\right]} \tag{435}$$

These relations are extremely useful for practical considerations of blast at moderate to long distances where the similarity solutions fail.

When we substitute the semi-empirical shape of pulse (403) into Eq. (416) we get for the positive impulse:

$$I = (p_2 - p_1)\, L'\, e^{-1}/c_1 \tag{436}$$

The "effective length" of the pulse is defined as $I\, c_1/(p_2 - p_1)$. Thus the "effective length" of the pulse is $L'/e = .368\, L$.

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